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Developing a new formulation and exact solution approach for continuous-time optimal control model: application to coordinating a supplier-manufacturer supply chain

Atefeh Hasan-Zadeh [1*]

[1] *Fouman Faculty of Engineering, College of Engineering, University of Tehran, Fouman, Iran.*

*** Corresponding Author:** *Atefeh Hasan-Zadeh (Email: hasanzadeh.a@ut.ac.ir)*

Abstract *–Supply chain coordination deals with the joint efforts of supply chain parties and making optimal global decisions, which in turn can improve the overall performance and efficiency of the entire supply chain. In many cases, the supply chain coordination problem leads to the formulation of a continuous-time optimal control model, where the optimal response is often calculated from numerical methods. Therefore, in this paper, a novel approach to optimal control problems is proposed by expanding a modern formulation supported by progressive concepts of differential geometry and Poisson geometry. This approach leads to providing an analytical answer to solve the optimal control problem in such a way that the Hamilton-Jacobi-Bellmann partial differential equation (PDE) can be converted into a reduced Hamiltonian system. To check the effectiveness of the proposed formulation, the problem of coordination of supplier development plans in a two-level supply chain including a single supplier and a manufacturing firm is investigated. The application of this approach is well illustrated by re-examining a numerical example. The unique advantages of the proposed approach lead to its efficiency in finding the exact solution of optimal control models in various optimization problems. The developed method provides further insights into analytical methods for solving supply chain coordination problems and is supported by advanced geometric concepts and structured instructions.*

Keywords– *Optimal Control Problem, Poisson Bracket, Hamiltonian System, Supply Chain Coordination, Supplier Development.*

I. INTRODUCTION

One of the main topics studied in supply chain management that has been paid attention to in recent years is supply chain coordination, which has received the attention and investigation of supply chain researchers and experts (Adabi & Mashreghi, 2019; Dastyar et al., 2020; Hasan-Zadeh, 2017). What is defined as supply chain coordination is the joint efforts of supply chain members who work together towards mutually defined goals and activities, including supplier development, coordination with suppliers and customers, etc. (Hasan-Zadeh, 2019; 2021a). It is concerned with making globally-optimal supply chain decisions that can benefit all supply chain members, instead of individual decisions (Hasan-Zadeh, 2021b; 2021c; Eshaghnezhad et al., 2023).

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The noteworthy point is that supply chain coordination undertakes a significant task in improving the overall performance of the supply chain and the lack of coordination between supply chain partners may reduce its efficiency and lead to adverse consequences in supply chain operations.Therefore, the centralized decision-making and various mechanisms are used by supply chain partners including revenue sharing, risk sharing, synchronized operation, etc., to achieve coordination purposes (Hasan-Zadeh, 2021b; Hasan-Zadeh, 2021c; Hasan-Zadeh & Mohammadi-khanaposhti, 2018; Hosseini-Motlagh et al., 2020a,b).

In many industries, manufacturing firms establish strategic and long-term relationships with their suppliers through the implementation and support of supplier development programs (Hosseini-Motlagh et al., 2019). The goal is to improve the performance and capabilities of the suppliers to meet short- and long-term supply needs of manufacturing firms, which in turn results in improving operational performance in terms of cost, quality, delivery, etc. (Hosseini-Motlagh et al., 2020; Hsieh, 2018; Hu et al., 2020; Jian et al., 2020; Arasteh, 2022; Javadi et al. 2022; Nabavi et al., 2021). Such a strong relationship between manufacturers and suppliers enhances the overall efficiency and profitability of both parties and helps to create a sustainable competitive advantage (Jian et al., 2020; Johari & Hosseini-Motlagh, 2020; Kotzab et al., 2019; Li et al., 2017).

Despite its potential benefits, supplier development programs may not be attractive to suppliers because suppliers may prioritize their own goals and be reluctant to modify their internal processes (Pham & Doan, 2020; Proch et al., 2017). Since the success of a supplier development program depends on mutual recognition and aligned objectives, coordination between supplier and manufacturer is required (Hosseini-Motlagh et al., 2020a; Kim, 2000; Kotzab et al., 2019; Jafarzadeh et al., 2022; Keshmiry Zadeh et al., 2021). Thus, the optimal decision on supplier development is characterized by a solution for the problem of supply chain coordination.

Many problems of supply chain coordination, which were mentioned above, involve formulating and solving a continuous time optimal control model with an equation of an incomplete Hamiltonian system, in which the exact optimal solution cannot be obtained, and instead, it should be approximately estimated by numerical analysis (e.g., Hosseini-Motlagh et al., 2019; Kim, 2000; Kotzab et al., 2019; Ivanov et al., 2016; Kar et al., 2015; Hsieh, 2018; Mohammed & Khudair, 2023; Sutrisno et al., 2022; Roth et al., 2023). Therefore, in this paper, a new approach supported by advanced concepts of differential geometry and Poisson geometry is used to provide an exact (and not approximate) answer. For this purpose, the main problem is transformed into a reduced Hamiltonian system, then its analytical answer is formulated and presented.

For this purpose, after studying some required concepts, it is possible to expand the optimal control problem by considering other derivatives of Hamilton's function. This, in turn, leads to the generation of a complete set of equations in $\mathbf{R}^m$ (in which the desired optimal control problem is considered by considering all variables and without replacing fixed values). For a Hamiltonian system, there is a relation between the symmetry groups of one-parameter variations of the system and the first integrals. On the other hand, the Hamilton-Jacobi-Bellmann PDE is formulated using dynamic programming, which in turn is an infinitesimal version of the optimal control problem as a partial differential equation (PDE). Then the Hamiltonian system related to this PDE can be considered. Finally, using the first integrals and extended symmetry groups, it can be demonstrated that the problem under consideration can be solved for any admissible control in a detracted Hamiltonian system corresponding to the Hamilton-Jacobi-Bellmann PDE in the control problem (nonlinear) solved.

The application of symmetry groups to detract the desired partial differential equation to a Hamiltonian system of ordinary differential equations is parallel to the procedures used for the Euler-Lagrange equations, with the advantage that it also includes transparent geometric interpretation. Thus, in this framework optimal control problems with dynamic requirement that apply Pontryagins maximum principle to the corresponding Lagrangian system (e.g., Bertsekas, 1999) can be investigated and generalized.

In this paper, the proposed approach mentioned above is used to obtain an optimal solution for the supplier development coordination problem in a two-level supply chain including a supplier and a single manufacturer. The remainder of this manuscript is formed as follows. In Section 2, the essential notions, and theorems on optimal control problems, differential geometry, and Poisson geometry are expressed, which are used to extend a novel demonstration of the model. Section 3 explains the ingredients of the suggested solution method. The proposed approach to the supplier development coordination problem in the supplier-manufacturer supply chain is discussed in Section 4. In section 5, a numerical example is presented to compare the numerical answer and the analytical answer derived from the proposed approach and to justify the various advantages of the proposed approach. Finally, the conclusion is discussed in the 6th section of the manuscript.

II. PROBLEM DESCRIPTION AND FORMULATION

A. Preliminaries

A.1. Optimal control problem

The optimal control problem is considered in the following simple and usual way:

$$\begin{aligned} &\max J = \int_0^T I\left(t,u,v\right)dt, \\ &\textit{provided that } \dot{u} = \varphi\left(t,u,v\right), \\ &u\left(0\right) = a, u\left(\tau\right) \textit{ arbitrary}, \left(a, \tau \textit{ given}\right) \\ &\textit{and } v\left(t\right) \in V, \forall t \in \left[0,\tau\right] \end{aligned} \tag{1}$$

Where, V is some bounded control set including t, time, u, state, and v as control, respectively. Also, another variable $\mu = \mu\left(t\right)$ is the costate variable (or subsidiary variable), which appears in the solution process by the Hamiltonian function $\mathcal{H}$, which is defined as follows:

$$\mathcal{H}\left(t,u,v,\mu\right) = I\left(t,x,u\right) + \mu\left(t\right)\varphi\left(t,u,v\right)\cdot \tag{2}$$

The maximum principle conditions that can be considered for the problem in (1), with the Hamiltonian function introduced in (2), are:

$$\begin{aligned} &\max_v \mathcal{H}\left(t,u,v,\mu\right), \textit{ for all } t \in \left[0,\tau\right], \\ &\dot{u} = \frac{\partial \mathcal{H}}{\partial \mu}, \left(\textit{equation of motion for } u\right) \\ &\mu = -\frac{\partial \mathcal{H}}{\partial u}, \left(\textit{transversality condition}\right) \end{aligned} \tag{3}$$

which symbol $\max\limits_v \mathcal{H}$ demonstrated that the Hamiltonian should be maximized considering only as the selection variable v and then a Hamiltonian system is obtained.

Note that, exclusive focus will be given to maximization problems in control theory.. This leads to a more detailed and transparent expression of the requirements for optimization conditions under which a minimization problem is addressed, such as in a sequel, it can always be reformulated as a maximization problem by multiplying the objective function by a negative sign.

Now, some geometric notions are studied, which are needed in the sequel. More details can be found in the literature (Bertsekas, 1999). In the following sub-section, a summary is given for some notions from differential and Poisson geometry to establish the main results so that the paper contains all the required concepts.

B. Geometric ingredients

B.1. Preliminaries of differential geometry

Similar to the notions presented in the literature (Olver, 1993; Rudolph & Schmidt, 2013), a k-dimensional manifold is considered as a set $\mathcal{K}$, besides denumerable coordinate charts $A_r \subset \mathcal{K}$ and one–to–one local coordinate maps $\eta_r : A_r \to B_r$ onto connected open subsets $B_r \subset \mathbf{R}^k$, with convincing the addendum properties:

The coordinate charts envelope $\mathcal{K}$. On overlap of any pair of coordinate charts $A_r \cap A_s$, composite map $\eta_s \circ \eta_r^{-1} : \eta_r\left(A_r \cap A_s\right) \to \eta_s\left(A_r \cap A_s\right)$, is a smooth function. If $u \in A_r$ and $\tilde{u} \in A_s$ are different points of $\mathcal{K}$ then, there exist open subsets $C \subset B_r$, $\tilde{C} \subset B_s$, with $\eta_r(u) \in C$, $\eta_s(\tilde{u}) \in \tilde{C}$, and $\eta_r(C) \cap \eta_s(C)$ as empty sets.

Suppose that λ is a smooth curve on a manifold $\mathcal{K}$, parameterized by $\lambda : [a,b] \to \mathcal{K}$, where $[a,b]$ is a subinterval of $\mathbf{R}$. At each point of λ, the curve has a tangent vector $\dot{\lambda} = \frac{d\lambda}{dt} = \left(\dot{\lambda}_1, \ldots, \dot{\lambda}_m\right)$. Tangent space to $\mathcal{K}$ at u presented by $T\mathcal{K}\,|_u$ is an k–dimensional vector space, with a basis $\left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\right\}$ in the given local coordinates. The integral curve of a vector field $\vec{\omega}\,|_u \in T\mathcal{K}\,|_u$ expresses a smooth parametric curve $u = \lambda(t)$ whose tangent vector at any point is equal to the value of $\vec{\omega}$ at the same point $\lambda(t) = \vec{\omega}\,|_{\lambda(t)}$ for all t.

For a vector field $\vec{\omega}$, introduced by $\Omega(t,u)$ the parameterized maximal integral curve transient of u in $\mathcal{K}$, which is called the flow generated by $\vec{\omega}$ or a one–parameter group of transformations, and the vector field $\vec{\omega}$ is also named the infinitesimal generator of the action. Local one-parameter groups of infinitesimal transformations and generators are in one-to-one correspondence with each other.

For computing the one–parameter group produced by a given vector field $\vec{\omega}$, it is referred to exponentiation of the vector field $\exp(t\vec{\omega})u \equiv \Omega(t,u)$ so that, for all $u \in \Omega$

$$\frac{d}{dt}\left[\exp(t\vec{\omega})u\right] = \omega\,|_{\exp(t\vec{\omega})u}, \tag{4}$$

If $\vec{\omega} = \sum_i \sigma_i(u) \frac{\partial}{\partial \sigma_i}$ and $\varphi : \mathcal{K} \to \mathbf{R}$ is a smooth function, then applying the chain rule and (4), then:

$$\frac{d}{dt}\varphi\left(\exp(t\vec{\omega})u\right) = \sum_{i=1}^{m} \sigma\left(\exp(t\vec{\omega})u\right) \frac{\partial \varphi}{\partial u_i} \exp(t\vec{\omega})u = \vec{\omega}(\varphi)\left[\exp(t\vec{\omega})(u)\right]. \tag{5}$$

For a smooth real–valued function $\varphi(u)=\varphi(u_1,...,u_m)$ of m independent variables, there are $m_n \equiv \begin{pmatrix} m+n-1 \\ n \end{pmatrix}$ different n–th order partial derivatives of φ. Multi–index notation is used $\partial\varphi(u)=\dfrac{\partial^n\varphi(u)}{\partial u_{l_1}\partial u_{l_2}...\partial u_{l_n}}$ for these derivatives. More commonly, if $\varphi: D \to A$ is a smooth function from $D \approx \mathbf{R}^m$ to $A \approx \mathbf{R}^\ell$, then $\upsilon=\varphi(u)=(\varphi_1(u),...,\varphi_\ell(u))$, the $\ell.m_n$ numbers $\upsilon_P^r=\partial_P\varphi_r(u)$ are necessary to express all the different n-order derivatives of the components of φ at a point u. Let $A_n \equiv \mathbf{R}^{\ell.m_n}$ be the Euclidean space of this dimension with coordinates υ_P^r corresponding to $r=1,...,\ell$, and all multiple indices of $P=(p_1,...,p_n)$ of order n such that it is pl to introduce the above derivatives. Moreover, $A^{(j)}=A\times A_1\times...\times A_j$ is a Cartesian product space whose coordinates express all derivatives of functions $\upsilon=\varphi(u)$ from all rows 0 to j. A generic point in $A^{(j)}$ is marked by $\upsilon^{(j)}$. It is supposed that the j–th length of a smooth function $\upsilon=\varphi(u)$, $\varphi: D \to A$, i.e. $\upsilon^{(j)}=pro^{(j)}\varphi(u)$, introduced by the equations $\upsilon_P^r=\partial_P\varphi_r(u)$. In addition, a system of differential equations of order j in m independent variables and s dependent variables is investigated, stated as a system of equations $\Delta_\theta(u,\upsilon^j)=0$, for $\theta=1,...,i$, containing $u=(u_1,...,u_m)$, $\upsilon=(\upsilon_1,...,\upsilon_\ell)$ and the derivatives of υ with respect to u up to order j.

For a vector field $\vec{\omega}$ on $\mathcal{K}\subset D\times A$, j–th prolongation of $\vec{\omega}$, expressed by $pro^{(j)}\vec{\omega}$ will be a vector field on j–jet space $\mathcal{K}^{(j)}$ and is introduced to be infinitesimal generator of the corresponding prolonged one–parameter group $pro^{(j)}\left[\exp(t\vec{\omega})\right]=\dfrac{d}{dt}|_{t=0}\ pro^{(j)}\left[\exp(t\vec{\omega})(u,\upsilon^{(j)})\right]$, for any $(u,\upsilon^{(j)})\in\mathcal{K}^{(j)}$.

Now, suppose $G(u,\upsilon^{(j)})$ be a smooth function of u, υ and derivatives of υ up to order j are defined on an open subset $\mathcal{K}^{(j)}\subset D\times A^{(j)}$. The total derivative of G with respect to u_l is the unique smooth function $D_lG(u,pro^{(j+1)}\varphi(u))=\dfrac{\partial}{\partial u_l}\left\{G(u,pro^{(j)}\varphi(u))\right\}$ expressed on $\mathcal{K}^{(j+1)}$. It can be represented that for $G(u,\upsilon^{(j)})$, l–th total derivative of G has a common form of

$$D_lG=\frac{\partial G}{\partial u_l}+\sum_{r=1}^{\ell}\sum_{P}\upsilon_{P,l}^r\frac{\partial G}{\partial \upsilon_P^r}, \tag{6}$$

Where, for $P=(p_1,...,p_n)$,

$$\upsilon_{P,l}^r=\frac{\partial\upsilon_P^r}{\partial u_l}=\frac{\partial^{n+1}\upsilon^r}{\partial u_l\partial u_{p_1}...\partial x_{p_n}}. \tag{7}$$

In (6), the sum is over all P's of order $0\le\#P\le j$, where n is the highest order derivative becoming in G. For a vector field $\vec{\omega}=\sum_{l=1}^{m}\sigma_l(u,\upsilon)\dfrac{\partial}{\partial u_l}+\sum_{r=1}^{\ell}\eta_r(u,\upsilon)\dfrac{\partial}{\partial\upsilon_r}$ on an open subset $\mathcal{K}\subset D\times A$, j–th prolongation of $\vec{\omega}$ is the vector field $pro^{(j)}\vec{\omega} = \vec{\omega} + \sum_{r=1}^{\ell}\sum_{P}\eta_r^P\left(u,\upsilon^{(j)}\right)\frac{\partial}{\partial\upsilon_P^r}$, stated on the corresponding jet space $\mathcal{K}^{(j)} \subset D \times A^{(j)}$. Coefficient functions η_r^P of $pro^{(j)}\vec{\omega}$ are admitting by the following formula: $\eta_r^P\left(u,\upsilon^{(j)}\right) = D_P\left(\eta_r - \sum_{l=1}^{m}\sigma_l\upsilon_l^r + \sum_{l=1}^{m}\sigma_l\upsilon_{P,l}^r\right)$, in which from (6) ,it will be $\upsilon_l^r = \frac{\partial\upsilon_r}{\partial u_l}$, and $\upsilon_{P,l}^r = \frac{\partial\upsilon_P^r}{\partial u_l}$.

A generalized vector field is stated as a (typical) formulation of the form $\vartheta = \sum_{l=1}^{m}\sigma_l\left[\upsilon\right]\frac{\partial}{\partial u_l} + \sum_{r=1}^{\ell}\eta_r\left[\upsilon\right]\frac{\partial}{\partial\upsilon_r}$, in which σ_l and η_r are also smooth differential functions, $\sigma_l\left[\upsilon\right] = \sigma_l\left(u,\upsilon^{(j)}\right)$ and $\eta_r\left[\upsilon\right] = \eta_r\left(u,\upsilon^{(r)}\right)$.

Note 2.1: It can be demonstrated that a generalized vector field $\vec{\omega}$ will represent a generalized infinitesimal symmetry of a system of differential equations in the following sense:

$$\Delta_\theta\left[\upsilon\right] = \Delta_\theta\left(u,\upsilon^{(j)}\right) = 0, \quad \theta = 1,\ldots,i, \tag{8}$$

if and only if

$$pro\vec{\omega}\left[\Delta d_\theta\right] = 0, \quad \theta = 1,\ldots,i, \tag{9}$$

for every smooth solution $\upsilon = \varphi(u)$.

B.2. Preliminaries of Poisson geometry

Similar to the notions introduced in the previous studies (Gutt, 2005; Da Silva & Weinstein, 1998), a manifold $\mathcal{K}$ has been considered with a Poisson structure $\{.,.\}$ on $\mathcal{K}$, meaning that to each pair of smooth real-valued functions R,S on $\mathcal{K}$, a smooth real-valued function $\{R,S\}$ on $\mathcal{K}$ is assigned with the main properties of bilinearity, skew–symmetry, Jacobi's identity and Leibniz's rule.

Consider $\mathcal{K}$ as a Poisson manifold on which $\mathcal{H}:\mathcal{K}\to\mathbf{R}$ is a smooth function. A Hamiltonian vector field corresponding to $\mathcal{H}$ is a unique smooth vector field $\hat{\omega}_{\mathcal{H}}$ on $\mathcal{K}$ satisfying $\hat{\omega}_{\mathcal{H}}(R) = \{R,\mathcal{H}\} = -\{\mathcal{H},R\}$. In the local coordinates $u = \left(u_1,\ldots,u_k\right)$ on $\mathcal{K}$, the Hamiltonian vector field will correspond to the general form of $\omega_{\mathcal{H}} = \sum_{l=1}^{m}\sigma_l(u)\frac{\partial}{\partial u_l}$, so that the coefficient functions of $\sigma_l(u)$ that depend on $\mathcal{H}$ are determined in a specific way.

Then,

$$\{R,\mathcal{H}\} = \sum_{l=1}^{k}\{u_i,\mathcal{H}\}\frac{\partial R}{\partial u_l}\cdot \tag{10}$$

Using skew–symmetry of the Poisson bracket and (10), the basic formula is obtained for:

$$\{R,\mathcal{H}\}=\sum_{l=1}^{k}\sum_{p=1}^{k}\{u_l,u_p\}\frac{\partial R}{\partial u_l}\frac{\partial \mathcal{H}}{\partial u_p}, \tag{11}$$

for the Poisson bracket. Structure functions of the Poisson manifold $\mathcal{K}$ relative to the given local coordinates, $M^{lp}(u)=\{u_l,u_p\}$ for $l,p=1,\ldots,k$ into a skew–symmetric $k\times k$ structure matrix $M(u)$ of $\mathcal{K}$ are assembled . Using (11), the Hamiltonian vector field associated with $\mathcal{H}(u)$ has the form of

$$\hat{\omega}_{\mathcal{H}}=\sum_{l=1}^{k}\left(\sum_{p=1}^{k}J^{lp}(u)\frac{\partial \mathcal{H}}{\partial u_p}\frac{\partial}{\partial u_l}\right) \tag{12}$$

Thus, in a specific coordinate chart, Hamilton's equations will be in the following form

$$\frac{du}{dt}=M(u)\,grad\,\mathcal{H}(u)\cdot \tag{13}$$

For example, in the manifold $\mathcal{K}=\mathbf{R}^{2j}$ with coordinates of $(x,y)=(x_1,\ldots,x_j,y_1,\ldots,y_j)$, if $R(x,y)$ and $S(x,y)$ are smooth functions, then their Poisson bracket is introduced as the following function:

$$\{R,S\}=\sum_{l=1}^{j}\left\{\frac{\partial R}{\partial y_l}\frac{\partial S}{\partial x_l}-\frac{\partial R}{\partial x_l}\frac{\partial S}{\partial y_l}\right\}. \tag{14}$$

In the case of standard bracket (14), as in Equation (11), the Hamiltonian vector field corresponding to $\mathcal{H}(x,y)$ is $\hat{\omega}_{\mathcal{H}}=\sum_{l=1}^{j}\left(\frac{\partial \mathcal{H}}{\partial x_l}\frac{\partial}{\partial y_l}-\frac{\partial \mathcal{H}}{\partial y_l}\frac{\partial}{\partial x_l}\right)$. The corresponding flow is achieved by integrating system of ordinary differential equations as follows

$$\frac{dy_l}{dt}=\frac{\partial \mathcal{H}}{\partial x_l},\quad \frac{dx_l}{dt}=-\frac{\partial \mathcal{H}}{\partial y_l},\quad l=1,\ldots,j, \tag{15}$$

which are the Hamilton's equations in this manner. More details can be found in the literature (Kotzab et al., 2019).

B.3. Main acquirements of Poisson and differential ingredients

According to Sections B.1 and B.2, it is concluded that:

Attainment 2.1: If $\hat{\omega}_f$ is the Hamiltonian vector field characterizing (12), using (5), for any solution to Hamiltonian equations $\frac{df(u(t),t)}{dt}=\frac{\partial f}{\partial t}(u(t),t)+\hat{\omega}_{\mathcal{H}}(f)(u(t),t)$, it will be $\frac{df}{dt}=0$ along solutions if and only if

$$\frac{\partial f}{\partial t}+\{f,\mathcal{H}\}=0, \tag{16}$$

holds everywhere. It follows that a function of $f(u,t)$ is a first integral for the Hamiltonian system (13) if and only if (16) preserves for all u,t.

Attainment 2.2: If $f(u,t)$ is the first integral of a Hamiltonian system, the Hamiltonian vector field $\hat{\omega}_f$ specified by f constructs a one-parameter symmetry group of the system, as in equation (12). This demonstrates the fact that the Hamiltonian vector field is an infinitesimal generator (in evolutionary form) of a one-parameter group of transformations acting on an open set of the space of independent and dependent variables for the system which is invariant under the element of the group. Then, applying (8) and (9) in Note 2.1, It can be determined that the vector field $\hat{\omega} = \omega_{\mathcal{H}} = f\left(t, u, \frac{\partial \upsilon}{\partial u}\partial_\upsilon\right)$ is the generalized symmetry of the Hamilton-Jacobi equation if and only if $f(t,x,y)$ is the first integral of Hamilton's equations.

Attainment 2.3: In a Hamiltonian system, one-parameter Hamiltonian symmetry groups whose infinitesimal generators are Hamiltonian vector fields are obtained from variational symmetry groups. What is the main topic of our methodology is the use of symmetry groups to reduce the order of a Hamiltonian system of ordinary differential equations. According to the previously presented concepts:

Theorem 2.1 (Reduction Theorem 1): Suppose that $\hat{\omega}_f \neq 0$ produces a Hamiltonian symmetry group of the Hamiltonian system $\dot{u} = Mgrad\,\mathcal{H}$ corresponding to the time–independent first integral $f(u)$. Then, there will be a reduced Hamiltonian system containing two fewer variables where every solution of the essential system can be specified by a single quadrature from those of the reduced system (Olver, 1993).

Theorem 2.2 (Reduction Theorem 2): Let $\dot{u} = Mgrad\,\mathcal{H}$ be a Hamiltonian system, which $g(u)$ does not depend on t. Then, there will be a reduced, time–dependent Hamiltonian system involving two fewer variables. Based on solutions of the detracted system, solutions of the primary system can be obtained by quadrature (Bertsekas, 1999).

III. PROPOSED METHODOLOGY

What is studied is the possibility of controlling the optimality of the response $\vec{u}(.)$ of the following ordinary differential equation (ODE):

$$\begin{aligned} &\dot{u}(t) = \varphi(\vec{u}(\varsigma), \vec{v}(\varsigma)), \quad (t < \varsigma < \tau) \\ &\vec{u}(t) = u, \end{aligned} \tag{17}$$

Where $\cdot = \frac{d}{dt}$, $\tau > 0$ is fixed final time, and $u \in \mathbf{R}^j$ is a given primary point attained by solution $\vec{u}(.)$ at start time $t \geq 0$. At subsequent times $t < \varsigma < \tau$, $\vec{u}(.)$ evolves according to an ODE, where $\varphi : \mathbf{R}^j \times U \to \mathbf{R}^j$ is a bounded, Lipchitz continuous function, and U represents some compact subset of $\mathbf{R}^k$. The function $\vec{v}(.)$ in (17) is a control, which contains of various appropriate schedules for moderating the parameters from the set U over time, thereby influencing the dynamics of the system formulated by (17). For details, see the literature (Evans, 1991).

Consider $\mathcal{U} := \{\vec{v} : [0,\tau] \to U \mid \vec{v}(.) \, is\ measurable\}$ as a set of acceptable controls. Here our objective is to detect a control $\vec{v}^*(.)$ that can drive the system optimally. To this end, for $u \in \mathbf{R}^j$ and $0 \le t \le \tau$, let us define the corresponding cost for each allowed control $\vec{v}(.) \in \mathcal{U}$:

$$\rho_{u,t}\left[\vec{v}(.)\right] := \int_t^\tau I\left(\vec{u}(\varsigma), \vec{v}(\varsigma)\right) d\varsigma + \xi\left(\vec{u}(\tau)\right), \tag{18}$$

where, $\vec{u}(.) = \vec{u}^{\vec{v}(.)}(.)$ solves the ODE (17) and $I : \mathbf{R}^j \times U \to \mathbf{R}$, $\xi : \mathbf{R}^j \to \mathbf{R}$ are certain functions, I is the running cost per unit of time and g is the final cost. Now, our essential consequences can be stated as:

Theorem 3.1 (Reduced Control Problem): The optimal problem (17) with the cost function (18) can be solved for any admissible control $\vec{v}(.) \in \mathcal{U}$, according to the above concepts, in a simpler way of a reduced Hamiltonian system with fewer variables.

Proof: **T**he dynamic programming method focuses on the value function $\upsilon(u,t) := \inf_{\vec{v}(.) \in \mathcal{U}} \psi_{u,t}\left[\vec{v}(.)\right], u \in \mathbf{R}^j$, $0 \le t \le \tau$ to investigate the above problem. For any a small enough $\delta > 0$ that $t + \delta \le \tau$, it will be:

$$\upsilon(u,t) = \inf_{\vec{v}(.) \in \mathcal{U}} \left\{ \int_t^{t+\delta} k\left(\vec{u}(\varsigma), \vec{v}(\varsigma)\right) d\varsigma + \upsilon\left(\vec{u}(t+\delta), t+\delta\right) \right\}, \tag{19}$$

where, $\vec{u}(.) = \vec{u}^{\vec{v}(.)}(.)$ solves the ODE (17) for the control $\vec{v}(.)$. Infinitesimal prescription of the optimality estates (19) can be inscribed as a PDE. The value function υ is the unique viscosity response of this ultimate-value problem for the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} &\upsilon_t + \min_{\alpha \in \mathcal{U}} \left\{ \varphi(u,\alpha).D\upsilon + I(u,\alpha) \right\} = 0, \ \ in \ \ \mathbf{R}^j \times (0,\tau) \\ &\upsilon = \xi \ on \ \ \mathbf{R}^j \times \{t = \tau\}, \end{aligned} \tag{20}$$

where, $D\upsilon = D_j \upsilon = \left(\upsilon_{u_1}, \dots, \upsilon_{u_j}\right)$. Same to problem (1) with the Hamiltonian system (2) in Section B, the Hamilton-Jacobi-Bellmann PDE (20) has the formation $\upsilon_t + \mathcal{H}\left(D\upsilon, u\right) = 0$ in $\mathbf{R}^j \times (0,\tau)$, for the Hamiltonian:

$$\mathcal{H}(x,u) := \min_{\alpha \in \mathcal{U}} \left\{ \varphi(u,\alpha).x + I(u,\alpha) \right\}, \tag{21}$$

for $x, u \in \mathbf{R}^j$, where x is the name of the variable which substitue the gradient $D\upsilon$ in the PDE. Corresponding to a Hamiltonian system, as in equation (15) the Hamilton-Jacobi PDE is $\frac{\partial \upsilon}{\partial t} + \mathcal{H}\left(\frac{\partial \upsilon}{\partial u}, u, t\right) = 0$. Note that the basic maximum caluse aforesaid in (3) is a simplified prescription of this equation. Now, the attainments 2.1-2.3 presented in Section B.3 can be applied step by step to this Hamiltonian system. Then, the order of the system can be detracted according to the reduction theorems 2.1 and 2.2 and the knowledge of first integrals. The structure of the proposed method is summarized in Figure 1.

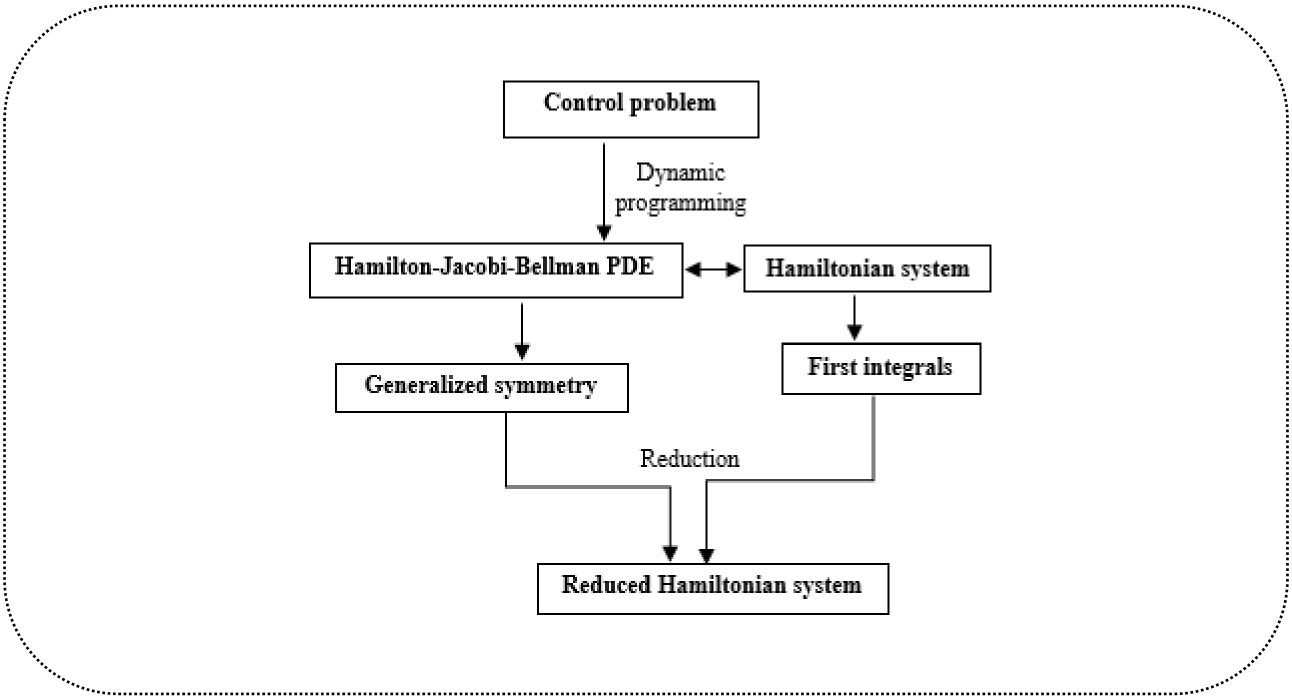

Fig. 1. Schematic of the exposed method

It is supposed that the Hamilton-Jacobi-Bellman equation of optimal control problem (17) in $\mathbf{R}^2 \times (0,\tau)$, as the $\upsilon_t + \min_{\alpha \in \mathcal{U}} \left\{ t\frac{\partial \upsilon}{\partial u_1}, t\frac{\partial \upsilon}{\partial u_2}.D\upsilon + \left(t(u_1,u_2)\right)^2 \right\} = 0$, for optional $\tau > 1$. Then, it will be:

$$\upsilon_t + \frac{1}{2}\left(\frac{\partial^2 \upsilon}{\partial u_1^2} + \frac{\partial^2 \upsilon}{\partial u_2^2}\right) + \left(u_1 - u_2\right)^2 = 0 \cdot \tag{22}$$

Step 1 (*Hamiltonian System):* Assume $\mathcal{K} = \mathbf{R}^4$ with standard Poisson bracket. Then, applying the relation (21) to (22), the corresponding Hamiltonian function is as follows.

$$\mathcal{H}\left(x_1, x_2, u_1, u_2\right) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + \left(u_1 - u_2\right)^2 \cdot \tag{23}$$

Based on Equation (2.15), the corresponding Hamiltonian system is expressed as follows.

$$\frac{du_1}{dt} = x_1,\ \frac{du_2}{dt} = x_2,\ \frac{dx_1}{dt} = -2\left(u_1 - u_2\right),\ \frac{dx_2}{dt} = 2\left(u_1 - u_2\right). \tag{24}$$

Step 2 (Symmetry Groups or First Integrals): According to Equation (23), it is discovered that the system receives an explicit translational invariance $\vec{\omega} = \partial_{u_1} + \partial_{u_2}$ and the corresponding first integral is $x_1 + x_2$.

Step 3 (Reduction): According to Theorem 3.1, the new coordinates $x = x_1 + x_2$, $u = u_1$, $w = x_1$, $z = u_1 - u_2$ is introduced which straighten out $\vec{\omega} = \partial_u$. In these variables, the Hamiltonian function is

$$\mathcal{H}\left(x,w,z\right)=w^2-pw+\frac{1}{2}x^2+z^2, \tag{25}$$

and $\left\{R,\mathcal{H}\right\}=\frac{\partial R}{\partial u}\frac{\partial \mathcal{H}}{\partial w}+\frac{\partial R}{\partial z}\frac{\partial \mathcal{H}}{\partial w}+\frac{\partial R}{\partial u}\frac{\partial \mathcal{H}}{\partial x}-\frac{\partial R}{\partial w}\frac{\partial \mathcal{H}}{\partial u}-\frac{\partial R}{\partial w}\frac{\partial \mathcal{H}}{\partial z}-\frac{\partial R}{\partial x}\frac{\partial \mathcal{H}}{\partial u}$. Furthermore, the Hamiltonian system is splitted into

$$\begin{aligned}
\frac{dx}{dt}&=-\frac{\partial \mathcal{H}}{\partial u}=0,\\
\frac{du}{dt}&=\frac{\partial \mathcal{H}}{\partial x}+\frac{\partial \mathcal{H}}{\partial w}=w,\\
\frac{dw}{dt}&=-\frac{\partial \mathcal{H}}{\partial u}-\frac{\partial \mathcal{H}}{\partial z}=-2z,\\
\frac{dz}{dt}&=\frac{\partial \mathcal{H}}{\partial w}=2w-x.
\end{aligned} \tag{26}$$

Step 4 (Solving the Reduced System): Response to the former couple, $x=c_1, u=\int w(t)dt+c_2$ (c_1,c_2 constant) can be achieved from solutions to the latter pair (26). They make a detracted Hamiltonian system respective to the reduced Poisson bracket $\left\{\tilde{R},\tilde{\mathcal{H}}\right\}=\tilde{R}_z\tilde{\mathcal{H}}_w-\tilde{R}_w\tilde{\mathcal{H}}_z$ for functions of w and z with the Hamiltonian system (25) produced by fixing $x=c_1$.

For obvious integration of Equation (26) and then, finding the precise answer of the primary system (24), again the proof of Theorem 3.1 is used such that, by setting $\mathcal{H}\left(w,z\right)=\beta+\left(\frac{1}{4}\right)x^2$, it is found that: $w^2-xw+\left(\frac{1}{2}\right)x^2+V\left(z\right)=\beta+\left(\frac{1}{4}\right)x^2$ or $w=\frac{1}{2}x\pm\sqrt{\beta^2-z^2}$. In this way, the solution is recovered just by integrating $\frac{dz}{dt}=2w-x\pm2\sqrt{\beta-z^2}$ and the exact solution of $\pm\int\frac{dz}{2\sqrt{\beta-z^2}}=\int dt$ or $w=\frac{1}{2}x\pm\sqrt{\left(t-\psi\right)^2-z^2}$ is obtained .

IV. APPLICATION OF THE PROPOSED METHODOLOGY TO COORDINATING SUPPLIER DEVELOPMENT

A. Problem description and formulation

The problem of coordinating supplier development in a two-echelon supply chain as presented in the study by Proch et al.(2017) is considered (Adabi & Mashreghi, 2019). The supply chain comprises a single supplier and a single manufacturing firm where the manufacturer congregates components from the supplier and sells the final products to the market. The goal is to identify the optimal decision for supplier development investment.

A centralized decision-making procedure is investigated and the supply chain is assumed as an integrated system where all parameters containing the optimal measure of endeavor invested in supplier development are chosen simultaneously. The performance of the total system is ensured by this decision-making procedure and the optimal level of supplier development is chosen, in the sense that it maximizes the profit of the whole supply chain. The variables and parameters of this model are epitomized in Table 1.

Table 1. Parameters and decision variables, (proch et al., 2017).

Parameters/ Variables	Description
a	Prohibitory cost (e.g. maximum willingness to pay)
b	Worth elasticity of the merchandise
c_M	Manufacturer's unit production cost
c_{SD}	Supply cost per unit charged by the supplier
c_0	Supplier's unit production cost at the begininig of the contract period
$x(t)$	The measurement of the efforts invested in the supplier development
m	The supplier learning rate
$c_S(x)$ $c_S(x)=c_0x^m$ $m=\frac{\ln(\theta)}{\ln\chi};$ $\theta\in[0,1], \chi>1$	Supplier production cost
r	The supplier fixed profit margin
$u(t)$	The effort at time t
$\omega(t)$	Capicity limit of $u(t)$ (resource accessibility in terms of time, man strength or budget)

The profit function $J^{SC}: L^1([0,T),\mathbf{R})\to\mathbf{R}$ of the set of measurable functions and the model of efforts invested in supplier development are defined by the following problem:

$$J^{SC} := \int_0^T \frac{\left(a - c_M - c_0x^m(t)\right)^2 - r^2}{4b} - c_{SD}u(t)\,dt, \qquad (27)$$
$$\textit{subject to } \dot{x}=u;\ u:[0,T)\to[0,\omega),\ x(0)=x_0=1.$$

The centralized collaboration strategy should be determined such that, the accumulated profit function (27) is maximized. Using the maximum principle (3) applied to the optimal control problem (27) with the Hamiltonian function of

$$\mathcal{H}(t,x,u,\mu)=\frac{\left(a - c_M - c_0x^m(t)\right)^2 - r^2}{4b} - c_{SD}u(t)+\mu(t)u(t), \qquad (28)$$

switching time t^* can be obtained by the solution to $\frac{\partial\mathcal{H}}{\partial u}\left(x^*,u^*(t),\mu(t)\right)=-c_{SD}+\mu(t)=0$. Then, as investigated in a previous study by Adabi & Mashreghi (2019) , t^* is obtained by numerical analysis from the following equation:

$$\frac{mc_0\left(1+\omega t^*\right)^{m+1}\left(a - c_M - c_0\left(1+\omega t^*\right)^m\right)}{2b}\left(t^* - T\right)=c_{SD} \qquad (29)$$

More details on the above fromulation have been given in the previous study (Adabi & Mashreghi, 2019).

B. Conversion of the model based on the proposed methodology

The optimization problem given in Equation (27) is a common form in many problems of supply chain coordination. It results in an equation with different parameters for switching time (Equation (29)) and the optimal control function, which can be only evaluated by numerical estimation. In fact, there exists only one equation with different parameters (Equation (29)).

What is more important is the case where the Hamiltonian $\mathcal{H}$ is linear in the control u. In particular, one of the simple situations is when $\mathcal{H}$ is plotted against u as a straight line with a positive or negative slope, since the optimal control is always found at a boundary of u. Therefore, the only task is to determine this boundary. Moreover, this case shows how optimal control theory can easily guide and manage a complex situation in calculations.

This simple approach apparently results in the elimination of some equations of the Hamiltonian system in the mentioned coordination optimization problem. For example, because an accurate determination of the capacity limit $\omega = \omega(t)$ of $u(t)$ in the problem is not critical to our discussion, it is exogenously recognized to be practicable for the problem. However, given the proposed approach, all the functions and parameters in the system along with their actual effect are considered. Thus, it will be possible to incorporate more variables in the coordination optimization model. This can be implemented by considering some variables as multiple functions and then, the Hamiltonian function as a function of these variables and their derivatives. Hence, possibly nonlinear optimal control problems are addressed, resulting in systems of fully Hamiltonian equations where the equations are considered as variables. Then, these complicated systems can be converted into reduced Hamiltonian systems with exact solutions using Theorem 3.1. In this way, the exact solution to the original supplier-manufacturer coordination model can be obtained.

C. Solution method

According to Equation (27), the corresponding Hamiltonian function (28) can be rewritten as:

$$\mathcal{H} = \mathcal{H}(\mu,x,u,d) = d\left(p\left(d(t)\right) - c_M - c_{SC}\right) - c_{SD}u(t) + \mu(t)u(t), \tag{30}$$

with production quantity of $d(t) = \dfrac{a - c_M - c_{SC}}{2b}$ and price distribution of $p(d) = p(d(t)) = a - bd = a + c_M + c_{SC}$; $c_{SC} = r + c_0 x^m$.

Step 1 (*Hamiltonian System):* Based on Equation (15), the Hamiltonian system will be: $\dfrac{\partial \mathcal{H}}{\partial x} = \dfrac{-2mc_0x^{m-1}(t)\left(a - c_M - c_0x^m(t)\right)}{4b} = -\dfrac{d\mu}{dt}$, $\dfrac{\partial \mathcal{H}}{\partial d} = \dot{d}\left(a - bd - c_M - c_{SC}\right) - bd = -\dfrac{du}{dt}$, $\dfrac{\partial \mathcal{H}}{\partial \mu} = u = \dfrac{dx}{dt}$,

$\dfrac{\partial \mathcal{H}}{\partial u} = -c_{SD} + \mu = \dot{d}$ which can be written as follows

$$\mu(t) = \frac{mc_0x^{m-1}(t)\left(a - c_M - c_0x^m(t)\right)}{2b}, \tag{31}$$

$$\dot{d}(t) = c_{SD} + \mu(t), \tag{32}$$

$$\dot{u}(t) = \dot{d}\left(-a + bd + c_M + c_{SC}\right) + bd \cdot \tag{33}$$

Step 2 (First Integrals): According to Equation (31), it will be:

$$\begin{aligned}\mu(t) &= \mu\left(t^*\right) - \int_t^{t^*} \frac{mc_0 x^{m-1}(s)\left(a - c_M - c_0 x^m(s)\right)}{2b} ds \\ &= \mu\left(t^*\right) - \frac{mc_0}{2b}\left(a - c_M\right)\left(I_{m-1}\left(t^*\right) - I_{m-1}(t)\right) + \frac{mc_0^2}{2b}\left(I_{2m-1}\left(t^*\right) - I_{2m-1}(t)\right)\end{aligned} \tag{34}$$

where, $I_m(s) = \int_0^s x^m(k)\,dk$. Also, according to Equation (32), it will be:

$$d(t) = d\left(t^*\right) - \int_t^{t^*}\left(\lambda(s) - c_{SD}\right)ds = d\left(t^*\right) - \left(t - t^*\right)c_{SD} - \int_t^{t^*}\lambda(s)\,ds \cdot \tag{35}$$

Then, substituting this into Equation (33) results in:

$$\begin{aligned}u(t) &= u\left(t^*\right) + \frac{mc_0}{4b}(-a + c_M + r)\left(I_{m-1}\left(t^*\right) - I_{m-1}(t)\right) \\ &+ \frac{mc_0^2}{4b}\left(I_{2m-1}\left(t^*\right) - I_{2m-1}(t)\right) - \frac{1}{2}(a - c_M - r)\left(t^* - t\right) + \frac{c_0}{2}\left(I_m\left(t^*\right) - I_m(t)\right)\end{aligned} \tag{36}$$

Finally, for $x(t) = 1 + \omega t$, $t \in [0, t^*]$, as expressed in Equation (34), it can be concluded that

$$\mu(t) = \mu\left(t^*\right) - \frac{c_0\left(a - c_M\right)}{2b\omega}\left(\left(1 + \omega t^*\right)^m - (1 + \omega t)^m\right) + \frac{c_0{}^2}{4b\omega}\left(\left(1 + \omega t^*\right)^{2m} - (1 + \omega t)^{2m}\right) \tag{37}$$

In addition, based on Equation (35), it will be:

$$\begin{aligned}d(t) &= d\left(t^*\right) + \frac{c_0\left(a - c_M\right)}{2b\omega}\left(1 + \omega t^*\right)^m\left(t^* - t\right) - \frac{c_0\left(a - c_M\right)}{2b(m+1)\omega^2}\left(\left(1 + \omega t^*\right)^{m+1} - (1 + \omega t)^{m+1}\right) \\ &- \frac{c_0{}^2}{4b\omega}\left(\left(1 + \omega t^*\right)^{2m}\left(t^* - t\right)\right) + \frac{c_0{}^2}{4b(2m+1)\omega^2}\left(\left(1 + \omega t^*\right)^{2m+1} - (1 + \omega t)^{2m+1}\right)\end{aligned} \tag{38}$$

Finally, Equation (36) results in:

$$\begin{aligned}u(t) &= u\left(t^*\right) + \frac{c_0}{4b\omega}(-a + c_M + r)\left(\left(1 + \omega t^*\right)^m - (1 + \omega t)^m\right) \\ &+ \frac{mc_0{}^2}{8(m+1)\omega b}\left(\left(1 + \omega t^*\right)^{2m+2} - (1 + \omega t)^{2m+2}\right) - \frac{1}{2}(a - c_M - r)\left(t^* - t\right) \\ &+ \frac{c_0}{2\omega(m+1)}\left(\left(1 + \omega t^*\right)^{m+1} - (1 + \omega t)^{m+1}\right)\end{aligned} \tag{39}$$

Step 3 (Reduction): Following Step 2, it will be $\left(-c_{SD} + p\right)\frac{\partial \mathcal{H}}{\partial p} = u\frac{\partial \mathcal{H}}{\partial u}$. Then, its first integral is $u = \pm\sqrt{2p^2 - 2c_{SD}}$ and Equation (30) is reduced to $\mathcal{H}(p,x,d) = d\left(p\left(d(t)\right) - c_M - c_{SC}\right) \pm \sqrt{2p^2 - 2c_{SD}}\left(-c_{SD} + p\right)$

V. NUMERICAL EXAMPLE AND DISCUSSION

In this section, a numerical example is presented to better illustrate the application and advantages of the proposed method. The data of this example is taken from the study of Proch et al.(2017). The proposed approach is applied and the exact solution algorithm is presented in the research to obtain the results and compare them with the results obtained from the numerical estimation. It helps to validate and evaluate the performance and efficiency of the proposed algorithm and to analyze and compare the quality of the obtained solution against a reference solution. The parameters of the cited numerical example are given in Table 2.

Table 2 - Parameter values for numerical analysis (adopted from Proch et al., 2017])

T	a	b	c_M	c_0	r	c_{SD}	ω	m
60	200	0.01	70	100	15	100000	1	- 0.1

For numerical analysis of the problem using the given parameter values, from Equation (37), 5 obtain $\mu\left(t^*\right)=\mu(T)-\frac{c_0\left(a-c_M\right)}{2b\omega}\left(\left(1+\omega T\right)^m-\left(1+\omega t^*\right)^m\right)+\frac{c_0}{4b\omega}\left(\left(1+\omega T\right)^{2m}-\left(1+\omega t^*\right)^{2m}\right)$. Since $\mu\left(t^*\right)=c_{SD}$, it will be $100000=0-\frac{100(200-70)}{0.02}\left(\left(1+60\right)^{-0.1}\left(1+t^*\right)^{-0.1}\right)+\frac{100}{0.04}\left(\left(1+60\right)^{-0.2}-\left(1+t^*\right)^{-0.2}\right)$, resulting in $t^*=9.844$.

Substituting the identified value in Equation (37), it will be $\mu(t)=-25655-0650000\left(1+t\right)^{-0.1}-250000\left(1+t\right)^{-0.2}$

Since $d\left(t^*\right)=\frac{\left(a-c_M-\left(rc_0x^{*m}\right)\right)}{2b}=-19348.65$, then based on Equation (38), it's concluded that $d(t)=-19348.65+512146.73\left(9.844-t\right)-$

$722222.22\left(8.54-\left(1+t\right)^{0.9}-155203.71\left(9.844-t\right)+312500\left(6.73-\left(1+t\right)^{0.8}\right)\right)$. Also, according to Equation (4.13),

it is obtained that $u(t)=u\left(t^*\right)-287500\left(0.78-\left(1+t\right)^{-0.1}\right)+13888.88\left(73-\left(1+t\right)^{1.8}\right)-57.5\left(9.88-t\right)$

$+55.55\left(8.54-\left(1+t\right)^{0.9}\right)$. The approximate value of t^* is equal to 9.212, as obtained numerically in the study by Proch et al.(2017) (Adabi & Mashreghi, 2019). However, the analytical solution algorithm developed herein provides a better answer as it yields a bigger objective value. The difference between the results is due to elimination of some equations of the Hamiltonian system, which is also a prevalent practice to find the answer to the optimal control model in coordination optimization problems.

Based on modeling and arguments given in Section 4 and the numerical example presented in Section 6, some of the most important advantages of the proposed approach can be mentioned in the following:

- The exact optimal value of switching time was calculated analytically, instead of being approximately estimated.
- Using our proposed methodology, the value of switching t^* was obtained as 9.844, which is clearly better than the result obtained in the study by Proch et al.(2017) for the presented maximization control problem.
- In the previous works (e.g., Proch et al., 2017; Kim, 2000; Kotzab et al., 2019; Ivanov et al., 2016; Kar et al., 2015; Hsieh, 2018), the optimal solution has been identified by eliminating some critical equations. Thus, some important characteristics of the problem should be overlooked. In fact, an accurate determination of $\mu(t)$, $d(t)$ and $u(t)$

variables has been exogenously assessed to be feasible or they should be approximately identified. But in the proposed method, in actual inspection, the variables are considered as multiple functions and then, the Hamiltonian function as a function of these variables and their derivatives.

- Our novel reformulation and proposed solution methodology yields some possibly nonlinear optimal control problems resulting in the systems of fully Hamiltonian equations with equations as equal as variables. Then, as proved in our paper, these complicated systems can be converted into the reduced Hamiltonian systems with exact solutions using strong geometric ingredients.
- Since $x(t)=1+\omega t$, the amount of $\mu(t)$, $d(t)$, and $u(t)$, as given respectively in Equations (37-39) was expressed in t^* as the exponential relations. Therefore, in comparison with the estimations presented in the study by Proch et al.(2017), better optimal value of switching time t^* clearly results in an effective increase in the whole benefit of the entire supply chain.

VI. CONCLUSIONS

In this paper, a novel approach for finding the analytical and accurate answer to the continuous time optimal control problem based on a new formulation of the progressive concepts of differential geometry and Poisson geometry was presented. For this purpose, geometric notions are applied about symmetric groups and first integrals to reduce the order of the Hamiltonian system. The proposed approach and solution method was applied to supply chain coordination problem in a two-echelon supply chain with the objective of finding the optimal decision of supplier development investment. The exact optimal solution and the optimum switching time for corresponding coordination problem with a single supplier and single manufacturing firm is obtained.

The main advantage of the proposed methodology is that it outperforms the numerical estimation approach which is prevalent in solving the optimal control models in coordination optimization problems. The proposed methodology converts the basic problem to the system of fully Hamiltonian equations with equations as equal as variables. It provides the analytic optimal solution and, thus, yields better results than those obtained through numerical estimation. What was presented in this article can be well implemented for other important optimization problems.

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