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**Pages: 1-8**

# Efficient Computation of the Free-Space Periodic Green's Functions for All Source-to-Observation-Point Distances

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***Abstract*— Surface integral equation formulations of periodic structures have received attention because of the inherent efficiency of surface unknowns and automatic satisfaction of radiation condition through the problem's Green's function. These formulations employ the periodic Green's function (PGF); the addition of potentials from all point sources as observed in the unit cell. Unfortunately, the resulting series (1) has slow convergence when direct summation (DS) is employed, which makes its usage in MoM codes rather costly. In this paper a new closed form is derived for efficient computation of the linear one-dimensional and planar (two-dimensional) periodic Green's function at small source to observation points' distances. When combined with an accelerated modal (Floquet-wave) expression for more distant observation points, an efficient form is obtained for all distances. The efficiency of the proposed formulations have been shown through numerical computation.**

***Index Terms*—** Computational electromagnetics, Green function, computationally efficient forms, Periodic structures.

## I. INTRODUCTION

Surface integral equation formulations of periodic structures have received attention because of the inherent efficiency; for the following reasons: Integral equation solvers are much more stable than differential equation solvers (e.g. in solvers such as the Method of Moment; MoM). Also, the number of unknowns and hence memory requirements are far less in surface-based integral equations because the unknowns are restricted to the surface as compared with the many volume unknowns in the

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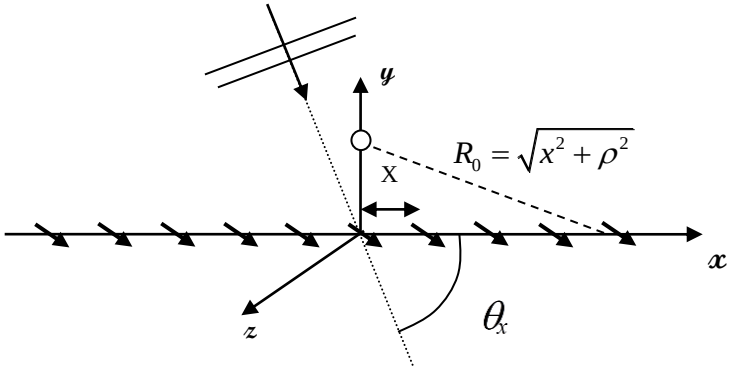


Fig.1. The assumed periodic array of point sources. The plane wave denotes graphically an inter-element delay of $\eta X = X \cos\theta_x$ .

$$G^{DS} = \sum_{m=-\infty}^{\infty} \frac{e^{-jk\left(R_m + \eta m X\right)}}{4\pi R_m} \tag{1}$$

$R_m = \sqrt{\left(x - mX\right)^2 + \rho^2}$,

conventional differential equation solvers (such as FDTD). Finally, automatic satisfaction of radiation conditions is guaranteed through the problem's Green's function [1-2] which alleviates the need for complicated boundary conditions; esp. for free space.

These formulations employ the periodic Green's function (PGF); the addition of potentials from all point sources as observed in the unit cell. Unfortunately, the resulting series (1) has slow convergence when direct summation (DS) is employed, which makes its usage in MoM codes rather costly.

Among the many acceleration techniques that have been applied to this problem [3] the modal (Floquet-Wave; FW) expression is a common practice. The PGF and the modal expression are given by (1) and (2) for a linear periodic array [3], with an inter-element phase increment of $kX\,\eta = kX\,\cos\theta_x$ ;

$$G^{FW}\left(\vec{r}, k\right) = \sum_{q=-Q}^{Q} \frac{e^{-jk_{xq}x}}{4jX} H_0^{(2)}\left(k_{\rho q}\rho\right), \tag{2}$$

$$k_{xq}\left(k\right) = k\eta + \alpha_q; \quad \alpha_q = \frac{2\pi q}{X}, \quad q = 0, \pm 1, \pm 2, \ldots$$

where $\rho = \sqrt{y^2 + z^2}$ and $k_{\rho q}\left(k\right) = \sqrt{k^2 - k_{xq}^2}$ under the condition $\mathrm{Im}\left[k_{\rho q}\right] \le 0$ due to the radiation condition at infinity. For $|q| > Q_1; Q_1 = \left\lceil X/\lambda . \max\left[\left(1-\eta\right), \left(1+\eta\right)\right]\right\rceil$ the modes are evanescent away from the array axis, which is why (2) is quite efficient [4] for observation points that lie sufficiently away ( $\rho > \lambda$ for $Q = Q_1$ ) from the array axis. Unfortunately, the modal expression (2) is not efficient for all

observation points, since it requires the summation of prohibitively many additional evanescent modes when the observation point lies in the proximity of array axis ($Q \approx nQ_1$ for $\rho = \lambda/n$) and even more so when approaching a source element. This adversely affects the accuracy of MoM matrix coefficients for self and near-self term interactions. The possibility of accelerating (2) near and on the array axis by means of repeated Shanks transform has been successfully shown [5], however, the computational cost for observation points near a point source still rises demandingly. This paper suggests a new accurate closed form for such special cases, thereby extending the applicability of the method to all observation points. To the best of the authors' knowledge, such a formulation has not been investigated previously.

## II. ACCELERATED COMPUTATION OF THE LINEAR PGF

### *A. Formulation*

One way of overcoming the complication for closer (e.g. self and near self-term) interactions is to use the direct summation for *near* and *self-term* interactions, i.e. within a range $R_0 = \sqrt{x^2 + \rho^2} < L_{prox}$.

However, a closed form expression can be found analytically for such cases, which significantly increases self-term accuracy while reducing computational cost. To elaborate, consider the condition $kR_0 << 1$, $R_m \approx mX$ and (1) can be rewritten as:

$$G_{proxim} = \sum_{m=1}^{\infty} \frac{e^{-jkmX\ (1+\eta)}}{4\pi mX} + \frac{e^{-jkR_0}}{4\pi R_0} + \sum_{m=1}^{\infty} \frac{e^{-jkmX\ (1-\eta)}}{4\pi mX}. \tag{3}$$

This expression can be rewritten using the series solution $\ln\left(1-e^{-a}\right) = \sum_{m=1}^{\infty} \frac{-e^{-am}}{m}$, as in (4);

$$G_{proxim} = \frac{e^{-jk\,(R_0)}}{4\pi R_0} - \frac{1}{4\pi X} \ln \begin{Bmatrix} \left(1 - \exp\left[-jkX\ (1-\eta)\right]\right) \cdot \\ \left(1 - \exp\left[-jkX\ (1+\eta)\right]\right) \end{Bmatrix}. \tag{4}$$

The expression in (4) is the desired closed form, and can be used together with (2) to compute the periodic Green's function efficiently for all the source to observation points' distances. Specifically, the Shanks-accelerated version of (2) can be used for $R_0 > L_{proxim}$ and (4) can be used for $R_0 < L_{proxim}$. The parameter $L_{proxim}$ is to be adjusted for optimal efficiency. A maximum relative error of $\delta G < 10^{-4}$ is

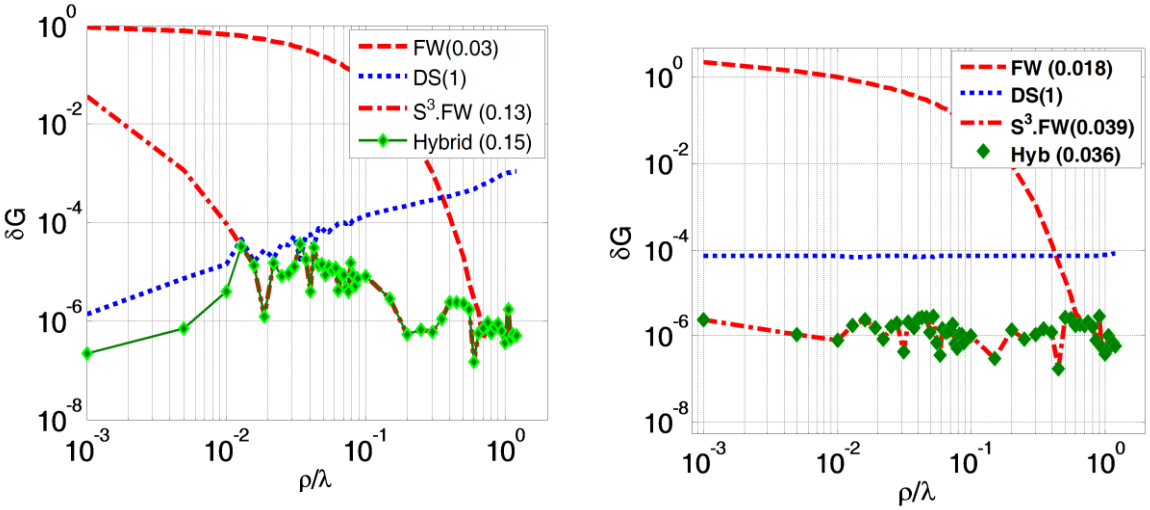


Fig. 2. Comparison of methods in terms of accuracy and computation time (normalized to that of DS): FW refers to (2) with $Q = Q_1 + 1$, DS refers to (1) with M=1000 (a) and 5000 (b). $S^3$.FW stands for triple shanks transform of the Floquet-waves expression, The Hybrid method assumes $L_{proxim} = \lambda/100$ and stands out in terms of efficiency. (a) First path with *x=0*, (b) second test path with *x=0.5*. The curves have been down sampled for clarity of the illustrations.

considered as sufficient for many applications [6].

### *B. Numerical Results*

To experimentally evaluate the performance of the proposed accelerated form for a comprehensive set of observation points, we consider:

$$\rho = \left\{ \begin{matrix} 10^{-3}\lambda, \\ [10^{-2} : 10^{-3} : 9\times 10^{-2}]\lambda, \\ [10^{-1} : 10^{-4} : 1]\lambda \end{matrix} \right\} \tag{5}$$

and *x=0* (First test path) or *x=X/2* (second test path). For each path, the number of points in every decade of (5) is approximately squared from the previous one, thereby simulating the distribution of source-observation points distances for a typical surface mesh. Consequently, the total computation time for all points in (5) can be taken as an estimation of matrix computation time in an actual Method of Moments problem. Figure 2 depicts the relative error versus distance and the total computation time of several approaches for an array with $X = 1.2\lambda, \eta = 0$. The exact evaluation of the PGF for error calculations has been performed using (1) with repeated Shanks transforms. It is observed that the proposed hybrid formulation achieves optimal performance for all distances with $L_{proxim} = \lambda/100$. For $\rho > \lambda/10$, no Shanks transform is needed if $Q = 10Q_1$ is chosen.

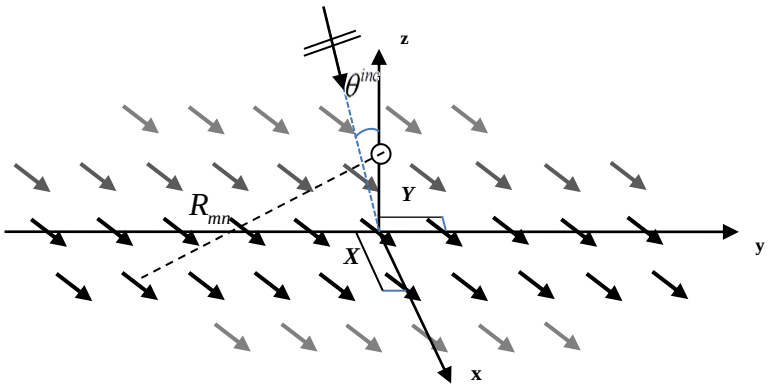


Fig. 3. Graphical representation of the planar array of point sources

## III. ACCELERATED COMPUTATION OF THE PLANAR 2D PGF

### *A. Formulation*

The assumed geometry for the array of point sources in this case is shown in Fig. 3.

Here again, we aim at developing the most efficient method of computation for all (near and far) source to observation distances. We will show that summation of Floquet modes is the most efficient for far distances, and for points near the source plane (xy-plane), interpolation over pre-stored tables of values of Green's functions would be efficient. We shall also describe the quantitative meaning of near and far distances.

Four methods are described and compared below: Direct summation, Floquet modes summation, and two novel combinations of the first two methods. For direct summation, we simply sum the potentials from all point sources at $\vec{L}_{mn} = mX\ \hat{x} + nY\ \hat{y}$ in Fig. 3, or

$$\tilde{G}\left(\vec{r},\omega\right) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{\exp\left[-jk\left(R_{mn} + mX\,\eta\cos\varphi^{inc} + nY\,\eta\sin\varphi^{inc}\right)\right]}{4\pi R_{mn}} \tag{6}$$

This performs best (converges with a small number of summations) at small source-observation distances, by increasing distance (z) its efficiency reduces drastically since all sources placed on any circle around the observation on the source plane can contribute almost equal amplitudes to the observed potential. Nevertheless, the convergence and accuracy of the direct summation method can be acceptably improved using conventional acceleration techniques such as the shanks transform. Here we have first converted the two-dimensional summation to a one-dimensional summation by summing over sources lying on squares around the observation point on the source plane. The results of this stage has been used to quantify the errors of the fast methods proposed.

The Floquet summation for a two-dimensional array leads to:

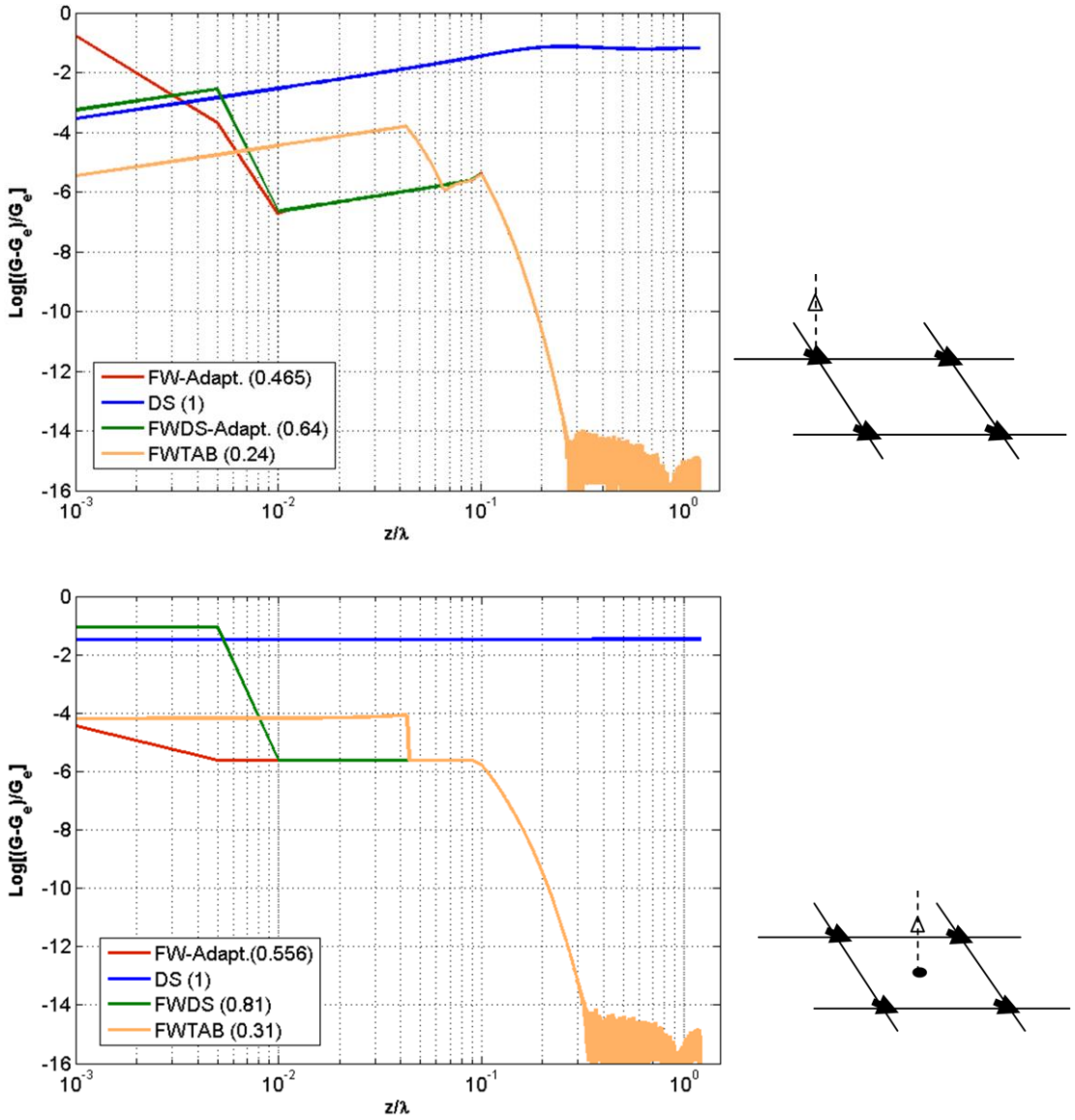


Fig. 4. Comparison of methods in terms of accuracy and computation time (normalized to that of DS): An array with $s_x = s_y = 0$ is assumed and period of X = Y = 1.2 wavelength. the DS results are obtained by taking 200 x 200 point sources into account. Points above (x = y = 0) (top) and above (x = 0.5X, y = 0.5Y) (bottom).

$$G = \sum_{p=-P}^{P} \sum_{q=-Q}^{Q} \frac{e^{-j\left(k_{xp}x + k_{yq}y + kzpq|z|\right)}}{2jXYk_{zpq}};$$

$$k_{xp} = s_x k + \frac{2\pi p}{X}$$

$$k_{yq} = s_y k + \frac{2\pi q}{Y}$$

$$k_{zpq} = \sqrt{k^2 - k_{xp}^2 - k_{yp}^2} \qquad (7)$$

This performs best at far source-observation distances.

### *B. Numerical Results*

For comparison of the four methods, we evaluate the PGF on two paths as shown in Fig. 4. Each path extends from z = 0 to z = 10λ. For an array with $X \approx \lambda$ and $s_x = s_y = 0$, Summation of one to several modes has satisfactory accuracy down to $z \approx \lambda$. Achieving the same accuracy down to z = 0.01λ requires a ten-fold increase in the number of modes, To use the Floquet modes summation adaptively (FW-Adapt in Fig. 4), we have segmented the z axis into the following intervals: (1λ, 10λ), (0.1λ,1λ), (0.01λ,0.1λ), (0.001λ, 0.01λ) and use 2,20,200 and 2000 modes fore them respectively.

For the third method (FWDS in Fig. 4), we combine DS and FW by using DS over z:(0.001λ, 0.1λ] with M = 1000 and FW over z : (0.1λ, 10λ] with 20 modes. The fourth method (FWTAB in Fig. 4) uses FW for far points and for near pints, the direct summation in (6) is decomposed into two parts: for source points nearer to the observation $M,N < N_{sep}$, (6) is used directly, for $M,N > N_{sep}$, (6) will lead to a smooth function in the unit cell around observation (above the origin in Fig. 3). This part is calculated once a'priori and stored as a table for sample points within the unit cell in the plane z = 0. This is the unit cell in which possible observation points will reside, or with slightly higher z. For every actual observation point (x, y, z), the evaluation of this second part is found by interpolation between samples in the table. Based on experience, 50 samples per wavelength can be sufficient in the unit cell in the z = 0 plane.

The parameters of the FWTAB method are selected as $N_{sep}$ = 15, $Q = P = 30X/\lambda$ and tables are made for M = 1409 and two consecutive shanks transformations.

## IV. CONCLUSION

We have presented a simple expression for accurate and efficient evaluation of one-dimensional periodic Green's function in the near-source region. Combined with an accelerated modal expression, it can provide for efficient computation of the periodic Green's function for all source to observation points' distances. For two-dimensional PGF, the far field is again computed by the Floquet wave summation. For the near field, the direct summation method is divided into two parts: point sources near the observation (which is almost on the plane z = 0) are computed by direct summation. Source points farther away result in a smoother function in the unit cell of observation. Consequently, their contribution can be tabulated on a set of sample points prior to the solution of the actual problem, and found by interpolation for every observation point afterwards. This method gives the fastest and most accurate results.

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