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# Optimal Relief Order Quantity under Stochastic Demand and Lead-time

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***Abstract*** – *In this paper, a newsboy model is developed under uniformly distributed lead-time and demand that is an appropriate assumption in obtaining optimal relief inventory of humanitarian disasters. It is noteworthy that limited historical data are in hand on relief operations. Hence, analytical and approximate solutions for optimal relief order quantity were derived. The effect of lead-time uncertainty on the optimal solution was analytically tested. The approximate solution was numerically evaluated and proper agreement with analytical data was achieved with a low variation coefficient of lead-time. The analytical results showed that lead-time uncertainty might increase or decrease relief order quantity, depending on the variation coefficient of lead-time.*

***Keywords –*** *Newsboy problem, Order quantity, Stochastic demand, Stochastic lead-time.*

## I. INTRODUCTION

Nowadays, humanitarian disasters such as large-scale earthquakes cause changes in demand and communication network disruption. On the other hand, existing models of Humanitarian Logistics (HL) originate from Commercial Logistics (CL), which are not practical for the real-world humanitarian phenomena. In this regard, recently, HL has drawn significant attention. Thomas (2008) studied HL and concluded that it had remained behind CL for thirty years. Whybark (2007) argued that HL had not been well studied and understood. According to an internal report of the International Federation of Red Cross and Red Crescent Societies (IFRC), the current lead-time during earthquake relief operation is unacceptably long.

Addressing issues related to lead-time uncertainty is challenging, especially in relief operation with long lead-time. The majority of the related research shows that a high variation of lead-time has negative effects on inventory management. For example, Chopra et al. (2004) studied the effect of lead-time uncertainty on safety stock and found that at cycle service levels above 50%, reduction in lead-time uncertainty would decrease the order quantity and safety stock. Moreover, reduction in lead-time variation was found more effective than that in mean due to the consequent dramatic decrease in the safety stock. Movahed & Zhang (2015) investigated the effect of lead-time uncertainty on optimal ordering as well as inventory policy and concluded that it would increase cost and order variation. Rahdar et al. (2018) proposed an inventory control model for uncapacitated warehouses in a manufacturing facility under demand and lead-time uncertainty to minimize total system cost.

The effect of lead-time uncertainty on optimal relief order in a newsboy problem is investigated in this paper. The main contributions of this study include the following:

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1) An HL is modeled under stochastic uniform demand and lead-time. The newsboy model is used to determine optimal relief order in order to minimize inventory cost. To our knowledge, this is the first research to obtain optimal relief inventory under stochastic demand and lead-time.

2) A closed-form solution for optimal order quantity is derived under uniform distribution of demand and lead-time. An approximate solution for optimal inventory, effective with a low coefficient of variation (the ratio of the standard deviation to the mean of a random variable), concerning stochastic lead-time ($CV_l$) is also offered.

3) For managerial insights, in comparison with the existing studies, the advantage of stochastic lead-time in optimal inventory is shown. Lead-time uncertainty results in decreased or increased inventory such that the inventory under stochastic lead-time is less than that under constant lead-time when $CV_l$ is less than a given threshold and vice versa.

The remainder of this paper is organized as follows. The next section contains a review of the related literature. In Section III, the newsboy problem under stochastic demand and lead-time is introduced. A closed-form solution for optimal order quantity is derived with uniform distribution of lead-time and demand. Section IV features an approximate solution for optimal order quantity using a triangular approximation. Numerical studies are carried out in Section V. Finally, in Section VI, a summary of the research is presented and future studies are suggested.

## II. LITERATURE REVIEW

A significant body of literature exists associated with disaster operation management. Galindo & Batta (2013) performed an extensive review of recent developments in the field of disaster operation management. Samani et al. (2018) proposed a multi-objective mixed-integer linear programming model of designing an integrated blood supply chain network for disaster relief under uncertain demand for blood products. Cao et al. (2018) introduced a multi-objective programming model of relief distribution for a sustainable disaster supply chain to reduce victims suffering under uncertain demand. Altay & Green III (2006) provided a comprehensive review of mathematical inventory modeling in disaster inventory management. Song et al. (2018) studied supply chain operations for rescue kits in disaster reliefs in a real-world application to minimize the total and peak tardiness of product delivery under demand uncertainty. Hu et al. (2017) presented a two-stage stochastic programming model for integrating decisions on pre-disaster inventory levels under uncertain demand and disaster type in humanitarian relief. Adhikary et al. (2019) studied a newsboy problem under bi-random demand in disaster relief operation to find the optimal order quantity that would maximize total expected profit. Although there is considerable research in the field of HL inventory management, e.g., Whybark (2007) and Kovács et al. (2009), few studies focus on inventory modeling in this context. On the other hand, most inventory models assume either demand or lead-time as a deterministic parameter. While studies on stochastic demand with constant lead-time are a popular research area in inventory modeling (Kouvelis et al., 2012; Moinzadeh & Nahmias, 1988), studies on constant demand with stochastic lead-time have received less attention (Zipkin, 1986).

Lead-time uncertainty, together with demand uncertainty, makes managing inventory more challenging and complicated for decision-makers (Chandra & Grabis, 2008; Hsieh, 2011; Pan et al., 2009). Pan et al. (2017) obtained optimal medical resource inventory for emergency preparation under uncertain demand and stochastic occurrence time considering different risk preferences at the airport. Many inventory optimization models ignore stochastic characteristics of production time and focus only on stochastic demand to hedge against demand variability (Birge & Louveaux, 2011; Dangl, 1999; Sting et al., 2014). Kamyabniya et al. (2017) presented a bi-objective location-allocation robust possibilistic programming model for designing a two-layer coordinated organization strategy in multi-type blood-derived platelets under demand uncertainty. Rodriguez et al. (2014) performed non-linear programming with stochastic demand to find the optimal inventory. Yongheng et al. (2014) developed a Lagrangean decomposition algorithm to decide on the optimal inventory in the electric motor industry under stochastic demand and constant lead-time. Kaya et al. (2014) developed a robust optimization method for optimal safety stock planning under stochastic demand and return in a closed-loop supply chain. Karimi & Ghodratname (2019) studied the effect of lead-time and demand uncertainty to determine optimal flexible and dedicated capacity in the newsboy problem and presented an approximation method to overcome complexities. Ho & Fang (2013) studied the inventory allocation of multiple products under uncertain demand and recommended that the inventory holding cost, the shortage cost, the loss of excess production, and market demands should be considered to discover the optimal inventory allocation concerning multiple products. Sun and Guo (2017) studied an inventory optimization of the newsboy model with fuzzy random demand to minimize total expected cost and used numerical methods to solve the problem. Adhikary et al. (2018) introduced a distribution-free newsboy problem in a fuzzy-random environment to obtain the optimal order quantity. They assumed a distribution-free model in which the mean and variance of the demand were known and the associated probability distribution was unknown.

In numerous studies, lead-time reduction has been viewed as an investment strategy (Bookbinder & Cakanyildirim, 1999). Christensen et al. (2007) conducted a survey involving manufacturers that consisted of 1264 individuals from the Institute of Supply Management. They concluded that lead-time variation had greater effects on the financial performance of a firm than on the lead-time mean and the average lead-times had no direct impact on the financial performance. He et al. (2011) also observed that lead-time uncertainty affected the cost and ordering policy more than the lead-time mean. Thorsen & Yao (2017) proposed a general methodology based on robust optimization for an inventory control problem subject to uncertain demands and uncertain lead-times in a finite horizon inventory problem.

Reviewing literature related to inventory management reveals that newsboy problems have attracted significant attention of members of the academic community. Ould-Louly & Dolgui (2002) studied the newsboy model in assembly systems in which demand was constant and lead-time was stochastic. They found that the optimally planned lead-time of the component could minimize total cost. Lodree et al. (2004) considered a newsboy model under stochastic demand to find the optimal order quantity and processing time by which response time and total expected cost could be minimized. Lodree et al. (2004) considered a newsboy problem under uncertain demand in which a Poisson distribution was followed. The adopted an approximation method to approximate the demand distribution. Dutta (2010) studied a newsboy problem under fuzzy demand and storage space constraint for the maximization of the total expected profit using a numerical algorithm to determine the optimal order quantity. Su & Pearn (2011, 2013) considered newsboy problems in which demand was assumed to follow a normal distribution. Huang (2013) developed a newsboy model under stochastic lognormal demand and constant lead-time for Poisson-type deterioration products. He discovered the optimal ordering policy that maximized the total expected profit. Similarly, Zhu et al. (2013) used a numerical algorithm for investigating an optimal newsboy problem with outsourcing, nonzero constant lead-time, and budget constraints to identify the optimal outsourcing policy. Kamburowski (2014) considered a distribution-free newsboy problem under worst- and best-case demand scenarios. Finally, Ding & Gao (2014) estimated demand based on uncertainty theory and found an optimal (a, S) policy for an uncertain multi-product newsboy problem to maximize expected profit. None of the previous studies in the literature has considered both demand and lead-time as uncertain parameters due to the high complexity of obtaining analytical results, especially for newsboy problems, in calculating product probability distribution function of lead-time and demand. On the other hand, considering both demand and lead-time as uncertain parameters leads to unexpected results. With this in mind, in this paper, it is attempted to find optimal relief order with a particular focus on large-scale earthquake disasters in which demand and lead-time are both stochastic.

## III. PROBLEM STATEMENT AND FORMULATION

HL is a branch of logistics that specializes in organizing the delivery and warehousing of supplies during natural disasters or complex emergencies to the affected area and people. In reality, it is far more complicated and includes forecasting and optimizing resources, managing inventory, and exchanging information. In comparison with CL, demand is highly variable in terms of timing, location, and products in HL. Also, lead-time is variable due to unknown locations, poor infrastructure, and unexpected events. The newsboy problem is a good choice to sustain reasonable inventory before the occurrence of any disaster. Therefore, a newsboy problem with stochastic demand and replenishment lead-time is considered in this paper. An order quantity is determined to satisfy the demand within the replenishment lead-time. It is assumed that both demand and lead-time follow uniform distribution. Once the demand and the lead-time are realized, an inventory cost occurs if the inventory level is more than the demand during the lead-time. Otherwise, penalty costs are incurred for unsatisfied demands. Schematic view of the problem is detailed as follows:

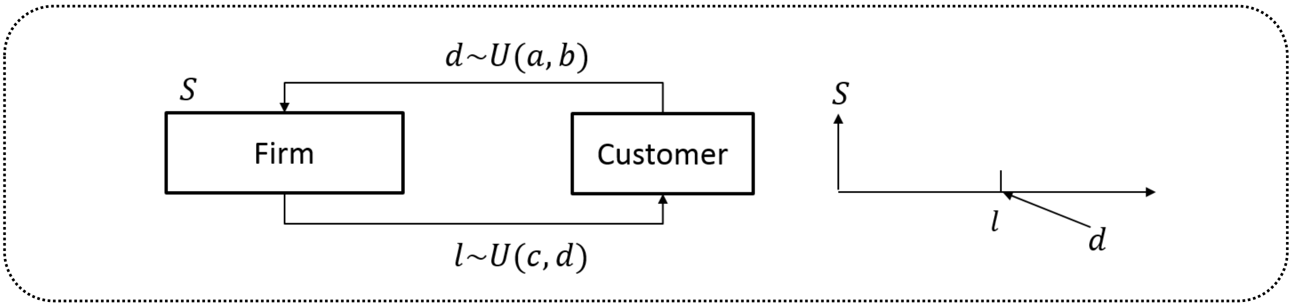


**Fig. 1. Schematics of the described problem**

Three demand distribution types have generally been used in literature, namely uniform distribution, normal distribution, and Poisson distribution (He et al., 2011). The Poisson distribution cannot represent relief demand characteristics since it calculates the gap between two discrete events. In contrast, the normal distribution requires a large amount of data to define its shape and parameters (mean and standard deviation). As there is limited historical data on HL, the normal distribution is not an appropriate analysis method for HLIM.

Given that, a uniformly distributed relief demand parameter is the most reasonable model to apply to the lead-time after an earthquake disaster. It is evident that the lead-time after a large-scale earthquake cannot be predicted quickly. However, uniform distribution parameters can be easily estimated by efficiently assessing local knowledge since the parameters are subjective estimates of the minimum and maximum values. Furthermore, it is assumed both demand and lead-time are stochastic, uniform, and independent.

Before developing the models, the notation utilized throughout the present paper is listed below:

## Parameters:

$D$ Stochastic demand for relief products, which follows a uniform distribution $D \sim U(a, b)$

$L$ Stochastic lead-time, which follows a uniform distribution $L \sim U(c, d)$

$L_c$ Constant lead-time

$p$ Selling price per unit of product

$w$ Unit production cost per unit of product

$h$ Inventory holding cost of excessive production per unit of product

$v$ Penalty cost per unit shortage of product

## Decision variables:

$S_s$ Order quantity under stochastic lead-time

$S_a$ Approximated order quantity under stochastic lead-time

$S_c$ Order quantity under constant lead-time

The expected profit function is formulated as follows:

$$\psi(S) = \int_0^S [px - h(S-x)] f_{DL}(x)dx + \int_S^\infty [pS - v(x-S)] f_{DL}(x)dx - wS, \tag{1}$$

where $f_{DL}(\cdot)$ is the Probability Density Function (PDF) of the demand during the lead-time. Using the Leibniz rule, the objective function (1) can be rewritten as:

$$\psi(S) = (p+v-w)S - v\mu_x - (p+h+v)\int_0^S F_{DL}(x)dx, \tag{2}$$

where $F_{DL}(\cdot)$ is the Cumulative Distribution Function (CDF) of the demand during the lead-time. Based on objective function (2), $\psi(S)$ is strictly concave on $S$ because:

$$\frac{\partial^2 \psi(S)}{\partial S^2} = -(p+h+v) f_{DL}(S) < 0. \tag{3}$$

Furthermore, the demand variation is assumed to be larger than the lead-time variation. PDF of the demand is obtained during the lead-time as follows (Glen et al., 2004):

$$f_{DL_s}(x) = \begin{cases} \dfrac{\ln x - \ln ca}{(b-a)(d-c)}, ca \le x < ad, \\ \dfrac{\ln d - \ln c}{(b-a)(d-c)}, ad \le x < bc, \\ \dfrac{\ln db - \ln x}{(b-a)(d-c)}, bc \le x < bd. \end{cases}$$

CDF of the demand during the lead-time is obtained as:

$$F_{DL_s}(x) = \begin{cases} \dfrac{ca + x(\ln x - \ln ca - 1)}{(b-a)(d-c)}, & ca \le x < ad, \quad (3a) \\ \dfrac{x(\ln d - \ln c) - a(d-c)}{(b-a)(d-c)}, & ad \le x < bc, \quad (3b) \\ \dfrac{x(\ln db - \ln x + 1) - a(d-c) - bc}{(b-a)(d-c)}, & bc \le x < bd. \quad (3c) \end{cases}$$

The lead-time follows a uniform distribution with a mean of $\mu = \dfrac{c+d}{2}$ and variation of $\sigma^2 = \dfrac{(d-c)^2}{12}$. Subsequently, $c = \mu - \sqrt{3}\sigma, d = \mu + \sqrt{3}\sigma$, and $CV_l = \dfrac{\sigma}{\mu}$. The optimal order quantity under stochastic and constant lead-times can be derived from Propositions 3.1 and 3.2, respectively.

**Proposition 3.1.** Under stochastic demand and lead-time, optimal order quantity can be achieved as follows:

$$S_s^* = \begin{cases} a\left(n-\sqrt{3}\sigma\right)e^{W_O\left(\frac{z-an}{ea\left(n-\sqrt{3}\sigma\right)}\right)+1}, & 0 \le \frac{\sigma}{\mu} < \frac{-1}{\sqrt{3}}\frac{W_O\left(-ue^{-u}\right)+u}{W_O\left(-ue^{-u}\right)-u}, \quad (4a) \\ \dfrac{z+a}{\ln\left(n+\sqrt{3}\sigma\right)-\ln\left(n-\sqrt{3}\sigma\right)}, & \frac{-1}{\sqrt{3}}\frac{W_O\left(-ue^{-u}\right)+u}{W_O\left(-ue^{-u}\right)-u} \le \frac{\sigma}{\mu} < \frac{-1}{\sqrt{3}}\frac{W_{-1}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)+\frac{au}{b}}{W_{-\iota}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)-\frac{au}{b}}, \quad (4b) \\ b\left(n+\sqrt{3}\sigma\right)e^{W_{-1}\left(\frac{-z+b-a-bn}{eb\left(n+\sqrt{3}\sigma\right)}\right)+1} & \frac{-1}{\sqrt{3}}\frac{W_{-1}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)+\frac{au}{b}}{W_{-\iota}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)-\frac{au}{b}} \le \frac{\sigma}{\mu}. \quad (4c) \end{cases}$$

where

$$u = \frac{a\left(h+w\right)+b\left(p+v-w\right)}{a\left(p+h+v\right)},$$

$$z = \sqrt{3}\sigma\left(u+\frac{\left(p+v-w\right)\left(b-a\right)}{p+h+v}\right).$$

$W_0(x), W_{-1}(x)$ are the values of the Lambert W function introduced by Corless (Corless et al., 1996), whose definition is provided in Appendix A.

*Proof.* Objective function (1) is concave in (3a), (3b), and (3c) because:

$$\frac{\partial\psi^2\left(S_s\right)}{\partial S_s^2} = \begin{cases} \dfrac{(p+v+h)(\ln ca-\ln S_s)}{(b-a)(d-c)} < 0, & ca \le S_s < ad, \\ \dfrac{(p+v+h)(\ln c-\ln d)}{(b-a)(d-c)} < 0, & ad \le S_s < bc, \\ \dfrac{(p+v+h)(\ln S_s-\ln db)}{(b-a)(d-c)} < 0, & b_C \le S_s < bd0, \end{cases}$$

Hence, the optimal solution for the objective function (1) in the given CDF piecewise using the first-order derivative from *Eq.* (2) is obtained.

$$\frac{\partial\psi\left(S_s\right)}{\partial S_s} = \left(p+v-w\right)-\left(p+h+v\right)F_{DL}\left(S_s\right)dx = 0,$$

$$S_s^* = F_{DL}^{-1}\left(\frac{p+v-w}{p+h+v}\right). \quad (5)$$

By substituting $F_{DL}$ in *Eq.* (5), according to *Eq.* (3a), the following is obtained:

$$S_s^* = a\left(n-\sqrt{3}\sigma\right)e^{W_0\left(\frac{z-an}{ea\left(n-\sqrt{3}\sigma\right)}\right)+1}. \tag{6}$$

$S_s^*$ in *Eq.* (6) is in $ac < S_s^* < ad.$ Then, by substituting $S_s^*$

$$0 \le \frac{\sigma}{\mu} < \frac{-1}{\sqrt{3}}\frac{W_0\left(-ue^{-u}\right)+u}{W_0\left(-ue^{-u}\right)-u}.$$

According to *Eq.* (3b), the expression below is derived.

$$S_s^* = \frac{z+a}{\ln\left(n+\sqrt{3}\sigma\right)-\ln\left(n-\sqrt{3}\sigma\right)}. \tag{7}$$

$S_s^*$ in *Eq.* (7) is in $ad \le S_s^* < bc.$ Then, by substituting $S_s^*$

$$\frac{-1}{\sqrt{3}}\frac{W_0\left(-ue^{-u}\right)+u}{W_0\left(-ue^{-u}\right)-u} \le \frac{\sigma}{\mu} < \frac{-1}{\sqrt{3}}\frac{W_{-1}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)+\frac{au}{b}}{W_{-1}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)-\frac{au}{b}}.$$

Based on *Eq.* (3c),

$$S_s^* = b\left(n-\sqrt{3}\sigma\right)e^{W_{-1}\left(\frac{-z+b-a-bn}{eb\left(n+\sqrt{3}\sigma\right)}\right)+1} \tag{8}$$

$S_s^*$ in *Eq.* (8) is in $bc \le S_s^* < bd.$ By substituting $S_s^*$, the following is obtained

$$\frac{-1}{\sqrt{3}}\frac{W_{-1}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)+\frac{au}{b}}{W_{-1}\left(-\frac{au}{b}e^{-\frac{au}{b}}\right)-\frac{au}{b}} \le \frac{\sigma}{\mu}.$$

When the lead-time variation is zero, that is, the lead-time is constant, it is set as the mean of the uniform lead-time $U(c,d)$ mentioned above equal to $L_c = (c+d)/2.$ Demand during the lead-time also follows a uniform distribution, i.e., $DL \sim U(a(c+d)/2, b(c+d)/2).$ Therefore, the PDF and CDF of the demand during the constant lead-time are:

$$f_{DL_c}(x) = \begin{cases} 0, & x < a\left(d+c\right)/2, \\ \frac{2}{(b-a)\left(d+c\right)}, & a\left(d+c\right)/2 \le x < b\left(d+c\right)/2, \\ 0, & b\left(d+c\right)/2 \le x. \end{cases}$$

$$F_{DL_c}(x)=\begin{cases}0, & x<a(d+c)/2,\\ \dfrac{2x-a(d+c)}{(b-a)(d+c)}, & a(d+c)/2\le x<b(d+c)/2,\\ 0, & b(d+c)/2\le x.\end{cases}$$

Under constant lead-time, the objective function (2) is formulated in the following quadratic form:

$$\psi_C(S_C)=A_C S_C^2+B_C S_C+C_C,$$

where:

$$A_c=\frac{-(p+h+v)}{(b-a)(d+c)},$$
$$B_c=(p+v-w)+\frac{a(p+h+v)}{b-a},$$
$$C_c=-\frac{a^2(d+c)(p+h+v)}{4(b-a)}-\frac{v(a+b)(c+d)}{4}.$$

Proof in proposition 4.1.

**Proposition 3.2.** Under constant lead-time, optimal order quantity and profit can be found as:

$$S_c^*=\frac{-B_c}{2A_c},$$

$$\psi_c^*(S_c^*)=\frac{-B_c^2}{4A_c}+C_c.$$

*Proof.* Note that $\psi_c(S_c)$ is concave in $S_c$ since $A_c$ is always negative. The optimal order quantity is obtained by setting its first-order derivative to zero. Then, the optimal profit can be obtained by substituting *S* in *Eq.* (2) with $S_c^*$.

Although lead-time uncertainty normally increases optimal order quantity, it is observed that order quantity under stochastic lead-time is lower than that under constant lead-time when $CV_l$ is smaller than a given threshold. Moreover, a high lead-time variation decreases order quantity under specific conditions.

**Proposition 3.3.** $S_s^*$ in Eq. (4b) is always smaller than $S_c^*$. That is, by substituting $c=\mu-\sqrt{3}\sigma$ and $d=\mu+\sqrt{3}\mu$ into Eq. (4b) and $S_c^*$, respectively, the following inequality is attained:

$$\frac{2\sqrt{3}\sigma\left((p+v-w)b+a(h+w)\right)}{(p+h+v)\left(\ln\left(n+\sqrt{3}\sigma\right)-\ln\left(n-\sqrt{3}\sigma\right)\right)}<\frac{n\left((p+v-w)b+a(h+w)\right)}{p+h+v}.$$

*Proof.* $S_s^*$ in *Eq.* (4b) is strictly decreasing with $\sigma > 0$ because $\frac{\partial S_s^*}{\partial \sigma} < 0.$ Thus, the maximum value of $S_s^*$ in *Eq.* (4b) can be obtained by:

$$\lim_{\sigma \to 0+} \frac{2\sqrt{3}\sigma\left(\left(p+v-w\right)b+a\left(h+w\right)\right)}{\left(p+h+v\right)\left(\ln\left(n+\sqrt{3}\sigma\right)-\ln\left(n-\sqrt{3}\sigma\right)\right)} = \frac{n\left(\left(p+v-w\right)b+a\left(h+w\right)\right)}{p+h+v}.$$

Therefore, $S_s^*$ in *Eq.* (4b) is always smaller than $S_c^*$.

**Proposition 3.4.** A threshold $\theta$ is:

$$\theta = -\frac{\left(\left(h+w\right)\left(b-a\right)+t\right)W_{-1}\left(\frac{-te^{\frac{(h+w)(b-a)-t}{(h+w)(b-a)+t}}}{b\left(p+h+v\right)}\right)+t}{\sqrt{3}t},$$

where

$$t = \left(-h+p+v-2w\right)b+2\left(h+w\right)a, \tag{9}$$

if $CV_l < \theta$, order quantity under stochastic lead-time is always lower than that under constant lead-time:

*Proof.* Based on Proposition 3.3, Proposition 3.4 can be proven by comparing $S_s^*$ in *Eq.* (4c) and $S_c^*$.

**Proposition 3.5.** A threshold $\beta$ is:

$$\beta = -\frac{1}{\sqrt{3}}\left(1+\frac{2\left(h+w\right)\left(b-a\right)}{t\,\ln\frac{t}{b\left(p+h+v\right)}}\right)$$

where t is defined in Eq. (9). If $CV_l < \beta$, optimal order quantity decreases by increasing $\sigma$. Otherwise, if $CV_l > \beta$, optimal order quantity increases by increasing $\sigma$.

*Proof.* The proposition can be shown by differentiating $S_s^*$ in *Eq.* (4a) with respect to $\sigma$.

## IV. APPROXIMATE SOLUTION

Although the optimal order quantity under stochastic lead-time can be obtained based on Proposition 3.1, the identification of the optimal order quantity is not very convenient, because evaluation of the *Lambert W* function value is not easy. Thus, an approximate solution for the optimal order quantity has been presented through a triangular approximation method by Areeratchakul & Abdel-Malek (2006), which can be used to approximate triangle-shaped CDFs, such as uniform, normal, and exponential distributions.

This method is used to first, estimate the area under the CDF function of demand during the lead-time. Based on (3a), (3b), and (3c), the following is obtained:

$$U(t)=\int_{ca}^{t} F_{DL}(x)\,dx \approx \frac{1}{2}\Delta(t-ca)^2, \tag{10}$$

where $\Delta$ is the slope of the approximated line obtained as (Areeratchakul & Abdel-Malek, 2006):

$$\Delta=\frac{0.9-0.001}{F_{DL_s}^{-1}(0.9)-F_{DL_s}^{-1}(0.001)}. \tag{11}$$

*Fig.* (2) shows the real (solid line) and the corresponding approximated distribution functions (dashed line) based on the sample data in Section V.

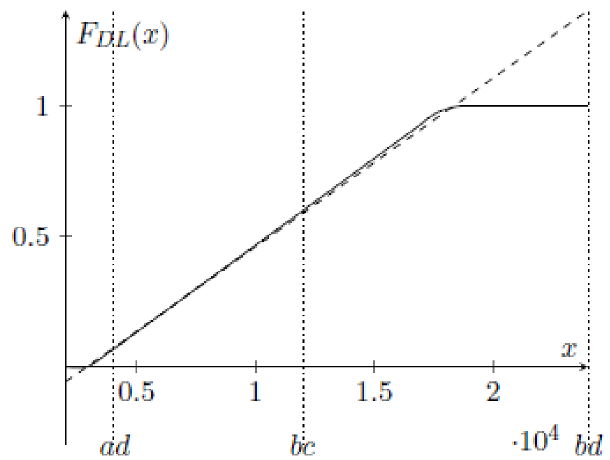


**Fig. 2. Estimated vs. original distribution functions for Problem No. 1 shown in Table I**

Using the triangular approximation, the approximation error of the area under the curve is estimated as:

$$e(t)=\frac{U(t)-\int_{ca}^{t}F_{DL_s}(x)\,dx}{\int_{ca}^{t}F_{DL_s}(x)\,dx} \tag{12}$$

*Fig.* (3) shows the approximation error using *Eq.* (12) when $t=bd$ .

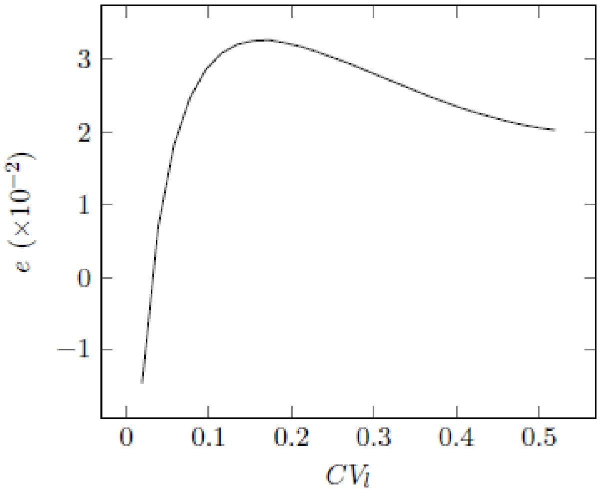


**Fig. 3. Approximation error of the area under the curve in *Fig.* (1)**

$S_s^*$ can then be approximated accurately based on Proposition 4.1.

**Proposition 4.1.** Optimal order quantity is approximated as:

$$S_s^* \approx S_a^* = -\frac{B_a}{2A_a},$$

*where*

$$A_a = \frac{-(p+h+v)\Delta}{2},$$
$$B_a = (p+v-w) + \Delta ac(p+h+v).$$

*Proof.* Given that the CDF of the demand during lead-time is triangular-based (see *Fig.* (1)), the integration of the CDF can be approximated via the triangular approach introduced by Areeratchakul & Abdel-Malek (2006). By substituting $t = S_a$ and $\mu_x = \frac{(a+b)(c+d)}{4}$ into *Eq.* (10), objective function (2) can be rewritten in the following quadratic form:

$$\psi_a(S_a) = A_a S_a^2 + B_a S_a + C_a,$$

where

$$A_a = \frac{-(p+h+v)\Delta}{2},$$
$$B_a = (p+v-w) + \Delta ac(p+h+v),$$
$$C_a = -\frac{\Delta a^2 c^2 (p+h+v)}{2} - \frac{v(a+b)(c+d)}{4}.$$

Also, $\Delta$ is the slope of the approximated line introduced in *Eq.* (11). With $\psi_a(S_a)$ as strictly concave in $S_a$ because $A_a < 0$, maximum $\psi_a(S_a)$ is obtained using its first-order derivation, thereby fulfilling the proof.

By utilizing the approximation method, it is not necessary to calculate the value of the *LambertW* function for obtaining optimal order quantity. While the approximation reduces the accuracy of the optimal order quantity, numerical analyses provided in the following section show that the approximation is effective when $CV_l$ is low.

## V. NUMERICAL EXPERIMENTS

In this section, a set of numerical experiments are conducted to demonstrate the influence of lead-time variation on the optimal order quantity. Problems reported in Table I are sorted based on $\sigma$.

**Table I. Sample data**

| Problem | a | *b* | $\sqrt{3}\sigma$ | $\mu$ | *c* | *d* | *p* | *v* | *h* | *w* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 600 | 6 | 30 | 24 | 36 | 200 | 20 | 30 | 30 |
| 2 | 100 | 600 | 7 | 30 | 23 | 37 | 200 | 20 | 30 | 30 |
| 3 | 100 | 600 | 8 | 30 | 22 | 38 | 200 | 20 | 30 | 30 |
| 4 | 100 | 600 | 9 | 30 | 21 | 39 | 200 | 20 | 30 | 30 |
| 5 | 100 | 600 | 10 | 30 | 20 | 40 | 200 | 20 | 30 | 30 |
| 6 | 100 | 600 | 11 | 30 | 19 | 41 | 200 | 20 | 30 | 30 |
| 7 | 100 | 600 | 12 | 30 | 18 | 42 | 200 | 20 | 30 | 30 |
| 8 | 100 | 600 | 13 | 30 | 17 | 43 | 200 | 20 | 30 | 30 |
| 9 | 100 | 600 | 14 | 30 | 16 | 44 | 200 | 20 | 30 | 30 |
| 10 | 100 | 600 | 15 | 30 | 15 | 45 | 200 | 20 | 30 | 30 |
| 11 | 100 | 600 | 16 | 30 | 14 | 46 | 200 | 20 | 30 | 30 |
| 12 | 100 | 600 | 17 | 30 | 13 | 47 | 200 | 20 | 30 | 30 |
| 13 | 100 | 600 | 18 | 30 | 12 | 48 | 200 | 20 | 30 | 30 |
| 14 | 100 | 600 | 19 | 30 | 11 | 49 | 200 | 20 | 30 | 30 |
| 15 | 100 | 600 | 20 | 30 | 10 | 50 | 200 | 20 | 30 | 30 |
| 16 | 100 | 600 | 21 | 30 | 9 | 51 | 200 | 20 | 30 | 30 |
| Problems are sorted in ascending order based on lead-time variation | | | | | | | | | | |

Using MATLAB software, results can be calculated as reported in Table II and illustrated in *Fig.* (3). In the sample data, the lead-time mean is $\mu = 30$ and the threshold $(\theta)$ is 0.222 based on Proposition 3.4. At $CV_l < \theta = 0.222$, the newsboy problem under stochastic lead-time identifies lower order quantity than that under constant lead-time. Therefore, lead-time variation does not necessarily increase the order quantity and it may either increase or decrease it.

**Table II. Profit and order quantity stock under stochastic and constant lead-times**

| Problem | $CV_l$ | S (constant LT) | S (stochastic LT) | $S_a^*$ (stochastic LT) | Profit (constant LT) | Profit (stochastic LT) |
|---|---|---|---|---|---|---|
| 1 | 0.1155 | 15000 | 14812.24 | 15286.07 | 1485000 | 1459759.4 |
| 2 | 0.1347 | 15000 | 14797.82 | 15417.44 | 1485000 | 1450837.9 |
| 3 | 0.154 | 15000 | 14810.25 | 15564.25 | 1485000 | 1441024.3 |
| 4 | 0.1732 | 15000 | 14843.78 | 15723.06 | 1485000 | 1430509.9 |
| 5 | 0.1925 | 15000 | 14894.3 | 15891.41 | 1485000 | 1419431.1 |

**Continue Table II. Profit and order quantity stock under stochastic and constant lead-times**

| Problem | $CV_l$ | S (constant LT) | S (stochastic LT) | $S_a^*$ (stochastic LT) | Profit (constant LT) | Profit (stochastic LT) |
|---|---|---|---|---|---|---|
| 6 | 0.2117 | 15000 | 14958.79 | 16067.51 | 1485000 | 1407888.5 |
| 7 | 0.2309** | 15000 | 15034.95 | 16249.97 | 1485000 | 1395958.6 |
| 8 | 0.2502 | 15000 | 15121 | 16437.72 | 1485000 | 1383700.9 |
| 9 | 0.2694 | 15000 | 15215.52 | 16629.9 | 1485000 | 1371162.6 |
| 10 | 0.2887 | 15000 | 15317.35 | 16825.8 | 1485000 | 1358381.6 |
| 11 | 0.3079 | 15000 | 15425.57 | 17024.85 | 1485000 | 1345389 |
| 12 | 0.3272 | 15000 | 15539.4 | 17226.55 | 1485000 | 1332210.6 |
| 13 | 0.3464 | 15000 | 15658.19 | 17430.49 | 1485000 | 1318867.8 |
| 14 | 0.3657 | 15000 | 15781.4 | 17636.28 | 1485000 | 1305378.9 |
| 15 | 0.3849 | 15000 | 15908.57 | 17843.6 | 1485000 | 1291759.2 |
| 16 | 0.4041 | 15000 | 16039.28 | 18052.13 | 1485000 | 1278022.1 |

**This value exceeds the threshold 0 = 0.222; hence, the order quantity under stochastic lead-time is larger than that under constant lead-time.

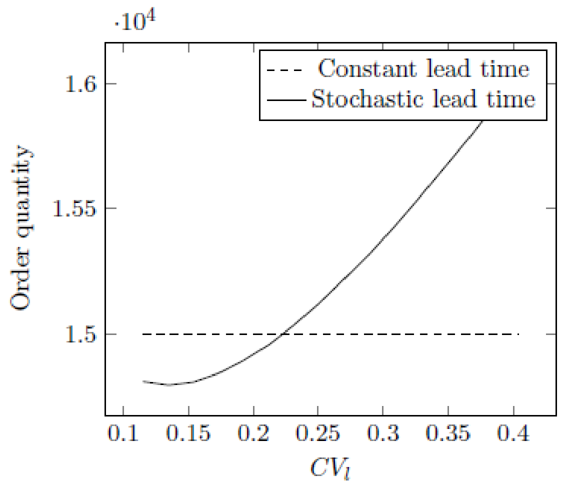


**Fig. 4. Comparison between order quantities under stochastic and constant lead-times based on $CV_l$**

*Fig.* (5) shows the profit under stochastic and constant lead-times. The former is found to constantly reduce the total profit of the firm.

*Figs.* (6-7) show approximated and exact optimal order quantities and approximation errors, respectively. The approximation error is estimated using $e = \dfrac{S_s^* - S_a^*}{S_s^*}$. As shown in the figures, the approximation method provides a reasonable estimation when $CV_l$ is low.

The results of the numerical example reveal that the introduced approximation method is useful when the coefficient of variation is low. On the other hand, it shows that optimal order quantity may decrease or increase by increasing CV, depending on the threshold value presented in proposition 3.4. This conclusion has never been discussed in the previous studies in the literature. It has always been concluded that order quantity increases on demand variation. It is suggested that this unexpected chaos is the effect of considering both demand and lead-time together as uncertain parameters.

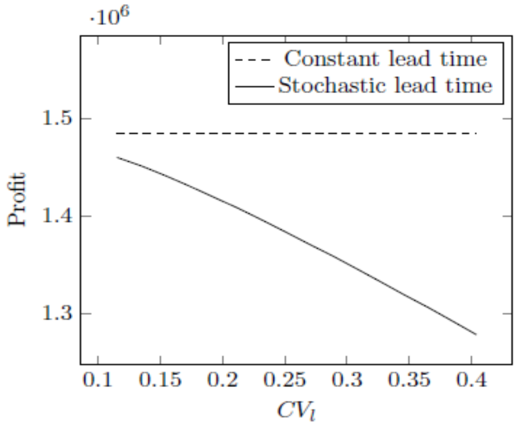


**Fig. 5. Comparison between profits under stochastic and constant lead-times based on $CV_l$**

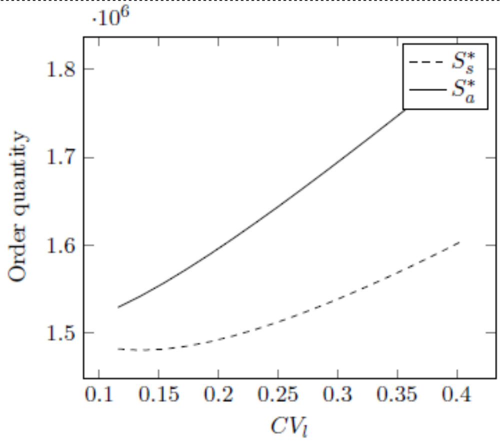


**Fig. 6. Comparison of order quantities using the approximated and exact solutions**

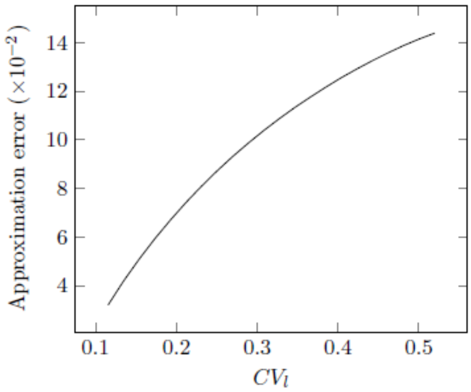


**Fig. 7. Order quantity approximation error based on $CV_l$**

## VI. CONCLUSION

This paper considers relief inventory modeling using a newsboy ordering policy under uncertainty in both demand and lead-time. Relief operations are highly challenging and diverse and require extensive effort. Using a uniform distribution for both demand and lead-time allows us to compute the probability distribution of the shortage and excess relief inventory. The model presented here was a stochastic optimization model based on first-order differential equations that attempted to determine the optimal order quantity. This necessitated preventing relief disruption for a given demand probability. This is the most appropriate way for decision-makers who do not know the actual lead-time demand curve. Exact and approximate solutions for optimal order quantities were derived. While high lead-time variation reduces HL performance and decreases total profit, it was found that it would decrease or increase the optimal order quantity, a finding never recorded in the previous literature. Moreover, a threshold associated with $CV_l$ was determined. When $CV_l$ was lower than the threshold, order quantity under stochastic lead-time was lower than that under constant lead-time.

Several future research directions are outlined in the following. In the proposed model, demand and lead-time are assumed to follow a uniform distribution. Therefore, considering other probability distributions such as normal distribution, which requires estimating the distribution parameters, is a major step towards developing more realistic models. Furthermore, while the approximation in this study is effective when *CVi* is low, more accurate estimations are needed to extend the proposed model with a high lead-time variation. The approximation of the *LambertW* function is also an interesting topic for future studies.

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## APPENDIX

*LambertW* is the solution to the following equation

$$x = W(x)e^{W(x)} \tag{A1}$$

This function is double-valued in $\frac{-1}{e} < x < 0$. $W_{-1}(x)$ is defined for $W \le -1$ and $W_0(x)$ for $W \ge -1$.

*Fig.* (A1) shows $W_0(x)$ and $W_{-1}(x)$ for $\frac{-1}{e} < x < 0$. Some values of $W_0(x)$ and $W_{-1}(x)$ are also provided in Table A1.

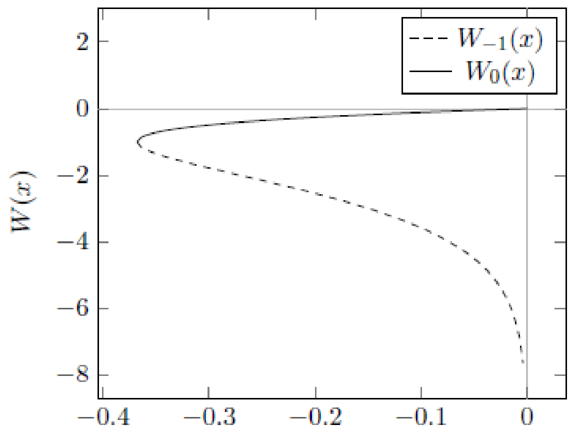


**Fig. A1. Function $W(x)$ for $\frac{-1}{e} < x < 0$**

**Table A1. Values of the *LambertW* function for $\frac{-1}{e} < x < 0$**

| $x$ | $W_0(x)$ | $W_{-1}(x)$ |
|---|---|---|
| -0.3679 | -1 | -1 |
| -0.3495 | -1.3554 | -0.713 |
| -0.3311 | -1.5318 | -0.6083 |
| -0.3127 | -1.6832 | -0.5327 |
| -0.2943 | -1.8244 | -0.4717 |
| -0.2759 | -1.9613 | -0.4199 |
| -0.2575 | -2.0973 | -0.3745 |
| -0.2391 | -2.235 | -0.3339 |
| -0.2207 | -2.3764 | -0.2971 |
| -0.2023 | -2.5235 | -0.2633 |
| -0.1839 | -2.6783 | -0.232 |
| -0.1655 | -2.8436 | -0.2028 |
| -0.1472 | -3.0223 | -0.1754 |
| -0.1288 | -3.2188 | -0.1495 |
| -0.1104 | -3.4392 | -0.1251 |
| -0.092 | -3.6926 | -0.1018 |
| -0.0736 | -3.9943 | -0.0797 |
| -0.0552 | -4.3724 | -0.0585 |
| -0.0368 | -4.8897 | -0.0382 |
| -0.0184 | -5.7439 | -0.0187 |
| 0 | 0 | $-\infty$ |